Week 15 Worksheet
Scattering

Jacob Erlikhman

4/26/23

Exercise 0. Warm Up. Consider the scattering of particles in central potentials.

a) Consider all particles with impact parameters between b and $b + \mathrm{d}b$. What is the cross-sectional area $\mathrm{d}\sigma$ of this region?

b) The particles from (a) are scattered with angles between θ and $\theta + \mathrm{d}\theta$. What is the solid angle $\mathrm{d}\Omega$ of the scattering region?

c) Using the previous two parts, show that the differential cross-section is

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{b}{\sin\theta}\left|\frac{\mathrm{d}b}{\mathrm{d}\theta}\right|. \tag{1}$$

Exercise 1. Small Angle Scattering. In this problem, you will calculate the differential scattering cross-section for small angle scattering. This regime applies when the impact parameter is large, so the force is small and the deflection angle is small. In this regime, we don't even have to go to the center of mass frame!

a) Suppose we are scattering particles of mass m in a region with a central potential $U(r)$. Take the x-axis to be the direction of the initial momentum of the scattered particle $p_0 = mv_0$. Denote by $\mathbf{p}'$ the final momentum. Remembering that we are in the small angle regime, what is the scattering angle θ in terms of $\mathbf{p}'$ and p_0?

   Hint: Think in terms of the components of $\mathbf{p}'$.

b) Use Newton's law to write $\dot{\mathbf{p}}'$ in terms of $U(r)$. Note that we are using cartesian coordinates for $\mathbf{p}'$ and spherical ones for $U(r)$!

c) Since the equation of (b) already contains the small quantity U, we may assume that the particle *is not deflected at all from its initial path*, i.e. that it moves in a straight line $y = b$ with uniform velocity v_0. Using (b) and this assumption, show that

$$p'_y = -\frac{b}{v_0}\int_{-\infty}^{\infty}\frac{\mathrm{d}U}{\mathrm{d}r}\frac{\mathrm{d}x}{r},$$

where b is the impact parameter and $\mathbf{r}$ is the location of the particle relative to the center of U. Suppose the scattering is contained in the (x, y)-plane, so this is a two-dimensional problem.

d) Change the integration variable from x to r, being careful with your limits of integration, and show that

$$\theta = -\frac{2b}{m{v_0}^2} \int_b^\infty \frac{\mathrm{d}U}{\mathrm{d}r} \frac{\mathrm{d}r}{\sqrt{r^2 - b^2}}.$$

Hint: If the particle moves in a straight line, what is r^2 in terms of x and b?

e) Finally, calculate the differential cross-section $\mathrm{d}\sigma/\,\mathrm{d}\Omega$ for scattering *in the lab frame* using your formula for θ above and Equation (1).