Week 2 Worksheet
Oscillators and Variational Calculus

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Exercise 1. Determine the oscillations of a spring-mass system under a force $F(t)$ of the following forms, if at time $t = 0$ the system is at rest in equilibrium (i.e. $x = 0$ and $\dot{x} = 0$).

Possible Hint: If you don't remember the analysis from class, you can do it quickly here. Make the substitution $\xi = \dot{x} + i\omega x$, where $\omega^2 = k/m$. Check that the equation of motion for x becomes $\dot{\xi} - i\omega\xi = F(t)/m$. If we find a solution for $\xi(t)$, then $x(t) = \mathrm{Im}[\xi(t)]/\omega$.

a) $F(t) \equiv F_0$ is a constant.

b) $F(t) = at$.

c) $F(t) = F_0 e^{-\alpha t}$.

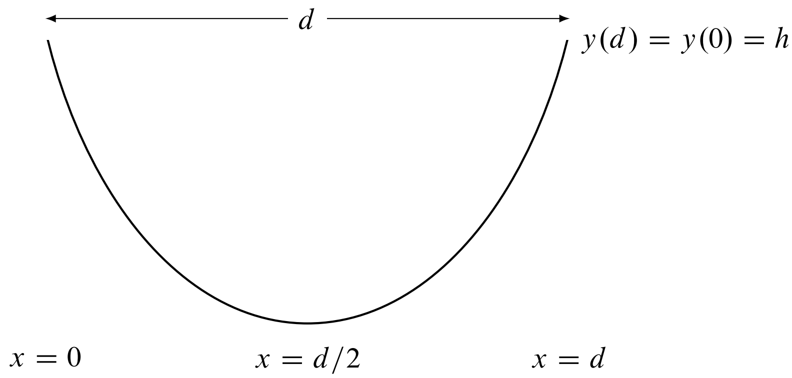
Exercise 2. Determine the final amplitude for the oscillations of a system under a force $F(t)$ which is 0 for $t < 0$, $F_0 t/T$ for $t \in (0, T)$, and F_0 for $t > T$. Suppose that up to $t = 0$ the system is at rest in equilibrium.

Exercise 3. In this exercise, you will find the equation of the **catenary**, which is the shape a chain takes when hung freely by its ends, as below.

a) Consider a chain as above of length ℓ suspended from two fixed points a distance d apart and of equal height above the ground. Suppose that the chain has uniform mass density ρ. What is the potential energy of the chain?

b) Minimize this energy, and use the Euler-Lagrange equation to find the differential equation for the function $y(x)$—the height of a point on the chain at a distance x on the horizontal, see the figure below. You should obtain

$$1 + y'^2 - yy'' = 0.$$



c) Solve the differential equation above using the steps below to obtain the solution

$$y(x) = C_1 \cosh\left(\frac{x + C_2}{C_1}\right).$$

1. Make the substitution

$$p = y' = \frac{\mathrm{d}y}{\mathrm{d}x}.$$

   Calculate $\mathrm{d}p/\,\mathrm{d}y$ to get everything in terms of $p, \mathrm{d}p/\,\mathrm{d}y$, and y
2. Separate variables, and integrate.
3. Replace back $p = \mathrm{d}y/\,\mathrm{d}x$, and separate variables again to solve for $y(x)$. You'll need the integral

$$\int \frac{\mathrm{d}t}{\sqrt{at^2 - 1}} = \cosh^{-1}(at)/a + C.$$

   The hyperbolic cosine function cosh is defined by $2\cosh(t) = e^t + e^{-t}$.

Remark. Actually determining the constants C_i is a bit beyond us. Here is the idea if you're interested (and/or want to try this at home). Write the length of the chain as

$$\ell = \int_0^d \mathrm{d}x \sqrt{1 + y'^2}.$$

Then, we can use the constraints $y(0) = h$ and $y(d) = h$ to hopefully get somewhere. Unfortunately, we need to solve a transcendental equation, and this isn't easy. If you're interested, bring this up during office hours, or see pp. 356-7 of Michael Spivak's *A Comprehensive Introduction to Differential Geometry Volume I, 3rd edition* (these pages are independent of the rest of the book).