Week 4 Worksheet
Central Forces and the Two-Body Problem

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Recall that the equation of motion for a central force F is given by

$$u''(\varphi) = -u(\varphi) - \frac{m}{\ell^2 u(\varphi)^2} F, \tag{1}$$

where $u = 1/r$ and ℓ is the angular momentum.

Exercise 1. Find the parametric solution to the equation of motion, i.e. $r(\varphi)$, for a particle that moves in a central potential $V(r) = -\alpha/r$.

Exercise 2. In this exercise, you will find the trajectory and the solution to the equation of motion for a particle of mass m in a "spherical well" potential, i.e. a potential of the form

$$V(r) = \begin{cases} -U_0, & r < R \\ 0, & r > R \end{cases}.$$

a) Explain why it would be difficult to solve Equation (1) using this potential.
Hint: Write the potential as a step function.

b) Explain why we can effectively consider this as a 2-dimensional problem, and write down the energy in polar coordinates, $r(t)$ and $\varphi(t)$.

c) If you haven't already done so, it is convenient to replace the term with $\dot{\varphi}$ by a term with the angular momentum, $\ell = mr^2\dot{\varphi}$. This should allow you to get an equation for $r(t)$ and its derivatives only.

d) Since there are no external forces, we can assume the total energy is constant. Thus, separate variables, and integrate. You should be able to solve this equation analytically for $r(t)$, though don't bother.

e) Now, rewrite the answer to (b) with $\mathrm{d}\varphi = \frac{\ell}{mr^2}\,\mathrm{d}t$. Separate variables again, and integrate from 0 to φ. You should obtain the intermediate solution

$$C = \varphi \mp \int \frac{\ell\,\mathrm{d}r}{r^2\sqrt{2m[E - U(r)] - \frac{\ell^2}{r^2}}},$$

where C is a constant. Solve this for $r(\varphi)$. The following integral will help

$$\int \frac{\mathrm{d}x}{\sqrt{ax^2-1}} = \frac{1}{\sqrt{a}} \tanh^{-1}\left(\frac{x\sqrt{a}}{\sqrt{ax^2-1}}\right).$$

f) Describe the trajectory of the particle; draw a picture. The identity $\sinh^2(x) - \cosh^2(x) = 1$ may help in simplifying the solution from (e).

Exercise 3. Suppose we drop a particle from a very tall height h towards the Earth. Ignoring air resistance, find the time T that it will take to reach the ground and the speed that it will obtain right before it hits the ground. You can leave your answer as an integral, but it can be solved analytically:

$$\int \frac{\mathrm{d}x}{\sqrt{a+1/x}} = \frac{x\sqrt{a+1/x}}{a} - \frac{\tanh^{-1}\left(\frac{\sqrt{a+1/x}}{\sqrt{a}}\right)}{a^{3/2}}.$$

Hint: Use the result of 2(d).

Remark. Note that the approach that you have covered in class lets us easily determine the Kepler orbits by analyzing the solution $r(\varphi)$. However, as Exercise 3 illustrates, we would sometimes like to know not only the trajectory, but the time that it takes for say an asteroid to hit the Earth. This is what the method developed in Exercise 2 provides us.