Week 7 Worksheet
Noninertial Reference Frames

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Exercise 1. *Quick Problems.*

a) Suppose you are in the backseat of an accelerating car holding a helium-filled balloon (on a string). Which way does the balloon tilt, and why?

b) A bird is flying due north at 45° latitude. In which direction does the Coriolis force act on the bird? Calculate the approximate magnitude of this force on the bird.

c) Does the direction that water drains (i.e. clockwise vs. counter-clockwise) change in the southern hemisphere vs. the northern hemisphere? Explain.
Hint: This is a trick question!

Remark. Hurricanes really do swirl in different directions depending on whether they are in the northern or southern hemisphere.

Exercise 2. Let R be the radius of the earth thought of as a perfect sphere and h the height of a tower from which an object falls at the equator (at the end, all equations can be multiplied by $\cos\lambda$ for the result at latitude λ). Let Ω be the angular velocity of the earth. Ignore air resistance.

a) Let $\omega(x)$ be the angular velocity of the object after it has fallen a vertical distance $x = \frac{1}{2}gt^2$. Show that

$$\omega(x) = \frac{\Omega(R+h)^2}{(R+h-x)^2}.$$

b) Show that in the approximation $h \ll R$, we have

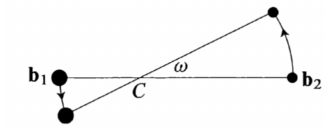
$$\omega(t) = \Omega(1 + gt^2/R).$$

c) If the time of descent is T, calculate the total angular displacement and the total displacement as measured in an inertial reference frame.

d) Calculate the displacement from the foot of the tower as measured along the earth's surface.

Exercise 3. *The restricted three-body problem, Lagrange points, and the Trojan Asteroids.*[1]
Suppose given 3 bodies, one of which has a negligible mass m compared to the other two, $\mathbf{b}_1$ and $\mathbf{b}_2$, with masses $m_1 > m_2$. Suppose further that $\mathbf{b}_1$ and $\mathbf{b}_2$ move in a plane and are in circular orbits about their center of mass C. Since the force on the third body depends on its distances from $\mathbf{b}_1$ and $\mathbf{b}_2$, the easiest coordinate system to work with is the noninertial one in which $\mathbf{b}_i$ are fixed in time. Choose the line from $\mathbf{b}_1(t)$ to $\mathbf{b}_2(t)$ to be the (rotating) x-axis, with C at $(0,0,0)$. Define $d_i = |\mathbf{b}_i|$.

[1] Adapted from Michael Spivak's *Physics for Mathematicians, Mechanics*, Addendum 10A.



a) Let a be the distance between $\mathbf{b}_1$ and $\mathbf{b}_2$. Write d_i in terms of it and the masses m_i.

b) Using you answer to (a) and Kepler's third law,

$$\tau^2 = \frac{4\pi^2 a^3}{G(m_1 + m_2)},$$

write the frequency of rotation ω of our rotating coordinate system in terms of d_1 or d_2.

c) Let coordinates for the third particle be given by $\mathbf{c}(t)$. Let $r_i = |\mathbf{c} - \mathbf{b}_i|$. Find the equation of motion for the third particle. It should have the form

$$\ddot{\mathbf{c}} = -G\left(m_1 \frac{\mathbf{c} - \mathbf{b}_1}{{r_1}^3} + m_2 \frac{\mathbf{c} - \mathbf{b}_2}{{r_2}^3}\right) + \mathbf{F}_{\text{cf}} + \mathbf{F}_{\text{cor}},$$

where $\mathbf{F}_{\text{cf}}$ and $\mathbf{F}_{\text{cor}}$ are the centrifugal and Coriolis forces, respectively.

d) Suppose we want to solve for stationary points of the orbit of the third mass *in the rotating coordinate system we are considering*, i.e. we want to solve for the case $\mathbf{c}(t) \equiv (C_1, C_2, C_3)$, where the C_i are constants. Starting with the EOM from (c), show that the solutions are in the configuration in the figure below: 3 colinear points at $x < -d_1$, $x \in (-d_1, d_2)$, and $x > d_2$, and 2 points that form an equilateral triangle between $\mathbf{b}_1$, $\mathbf{b}_2$, and the point.

Remark. Part (d) is by far the hardest part of the worksheet. The points you obtain are the Lagrange points $L_1, \ldots, L_5$. It turns out that only L_4 and L_5 are stable. The Trojan asteroids are those located at L_4 and L_5 for m_2 being Jupiter (there are thousands of these); there are also such asteroids for Mars and Neptune (though not Earth). Although the points L_1 and L_2 are unstable, they are used for satellites, such as the Solar and Heliospheric Observatory (SOHO) at L_1 and WMAP and the James Webb telescope at L_2. They are kept in place by onboard rockets.