Week 9 Worksheet
Rigid Bodies and Normal Modes

Jacob Erlikhman

3/20/23

Exercise 1. Consider a rectangular solid of three unequal dimensions, like a filled match box.

Remark. The results of this exercise apply equally well to any solid, e.g. a tennis racket, with three different moments of inertia $I_1 < I_2 < I_3$.

a) What are the principal axes?

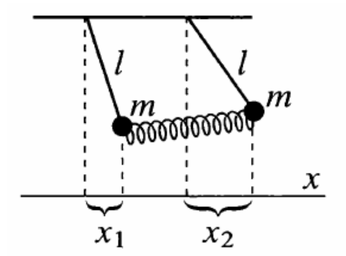
b) Experimentally, try to throw the box vertically and make it spin around each of these axes. Which of the axes from (a) work, and which don't? You should find that you can (with a deft throw) get the box to spin around two of the axes, but one of them will almost always result in an unwieldy tumbling motion.

c) Based on (b), make an argument for which axes are stable, and which aren't. Back up your experimental argument with a physical explanation, using the following steps.

   1. Use conservation of angular momentum and kinetic energy (why are these conserved?) to obtain two equations for the components of $\mathbf{L}$, $L_i = I_i \omega_i$.
   2. Notice that one of the equations is that of a sphere and the other that of an ellipsoid (in (L_1, L_2, L_3)-space). Simultaneous solutions of the equations will correspond to intersections of these surfaces. Graphically, find the forms of the intersections corresponding to rotations about each of the principal axes.
      Hint: To start, consider limiting cases. For example, the smallest sphere intersecting an ellipsoid (with the same center) will be the one that touches it in exactly two places: the endpoints of the minor axis.
   3. Make the argument for or against stability about each of the principal axes, $I_1 < I_2 < I_3$.

Exercise 2. Consider a coupled pendulum as in the figure below. To start, suppose equal masses and lengths. We could go through with a rigorous analysis of normal modes à la Taylor, but let's try to work intuitively.

a) Use physical intuition to find the types of motion that correspond to the normal modes.
   Hint: The normal modes have something to do with the masses moving "in sync."



b) Derive the equations of motion for both types of motion in (a). Let $\omega_0 = \sqrt{g/l}$ and $K = k/m$, where k is the spring constant.

c) Show that for one of the normal modes both pendulums have frequency ω_0 and for the other both pendulums have frequency

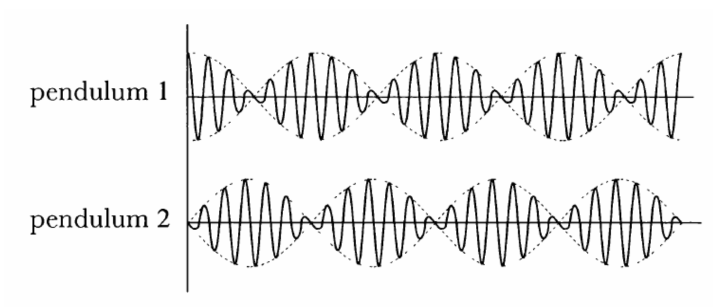
$$\omega = \sqrt{{\omega_0}^2 + 2K}.$$

Why is $\omega > \omega_0$?

d) Now, suppose we have arbitrary initial conditions. Using your results from (b) and (c), write down the general solutions for $x_1(t)$ and $x_2(t)$ (you don't have to solve for the four constants which necessarily appear).

e) Solve the equations for the initial conditions $x_1(0) = C$, $\dot{x}_i(0) = 0$, $x_2(0) = 0$.

f) In the situation of (e), argue that for weak coupling, i.e. $K/\omega_0 \ll 1$, we get beats:



Exercise 3. Suppose we are in the situation of Exercise 2, but now the pendulums have different lengths, l_1 and l_2, and different masses, m_1 and m_2. Again, let's try to work intuitively. Set

$$K_i = \frac{k}{m_i} \qquad \omega_i = \sqrt{\frac{g}{l_i}}.$$

a) Write down the general form of the equations of motion.

b) Based on your work in Exercise 2, argue why it is reasonable to look for solutions of the equations of motion that are complex exponentials *with the same frequency*.

c) Look for solutions that are the real part of complex solutions

$$x_1(t) = Ae^{i\lambda t}, \qquad x_2(t) = Be^{i\lambda t}.$$

d) Solve for λ, denote the solutions for λ as f_1 and f_2, and let B/A be C_1 and C_2 (since this will depend on λ).

e) Write down the solutions to the equations of motion, $x_i(t)$, in terms of C_i and f_i, time t, and 4 unknown constants, which depend on the initial conditions.

f) Determine the unknown constants for the same initial conditions as before, $x_1(0) = C$, $\dot{x}_i(0) = 0$, $x_2(0) = 0$.

g) Argue that for weak coupling, we still get beats, as in 2(f).

Exercise 4. Solve Exercises 2 and 3 (note that 2 is a special case of 3) using the method of finding the normal modes in Taylor.