Midterm Review Problems

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Exercise 1. A solid conducting sphere of radius a is in a constant, uniform external electric field $\mathbf{E_0}$. It is cut in half into two identical halves with an infinitely thin cut, which is perpendicular to $\mathbf{E_0}$. What force $\mathbf{F}$ acts on each half? How will this force change if we turn off the external field $\mathbf{E_0}$?

Let the external field be along the z-axis, i.e. $\mathbf{E_0} = E_0\hat{z}$, and suppose the conductor has potential $V = 0$ (note that we can't put the zero of the potential at infinity, since the electric field is not 0 there). The potential was worked out in Example 3.8 in Griffiths (though you should be able to do this on your own!)—it is given by

$$V(r,\theta) = -E_0\left(r - \frac{R^3}{r^2}\right)\cos\theta.$$

The charge density is

$$\begin{aligned}\sigma =& -\varepsilon_0\frac{\partial V}{\partial n}\\ =&\varepsilon_0\left(E_0 + \frac{2R^3}{R^3}\right)\cos\theta\\ =&3\varepsilon_0 E_0\cos\theta.\end{aligned}$$

since $n = r$. Now, let's calculate the pressure on one of the halves. It is given by

$$\frac{1}{2\varepsilon_0}\int \sigma^2\cos\theta\,\mathrm{d}a,$$

where the integral is taken over the surface of one of the halves and we have a $\cos\theta$ term to project onto the z-axis (since the other directions will cancel each other out). We can calculate this explicitly as

$$\frac{9\varepsilon_0{E_0}^2}{2}\int_0^{\pi/2}\mathrm{d}\theta\cos^3\theta\sin\theta\cdot 2\pi a^2 = -9\pi\varepsilon_0{E_0}^2a^2\frac{\cos^4\theta}{4}\bigg|_0^{\pi/2} = \frac{9\pi\varepsilon_0{E_0}^2a^2}{4}.$$

When we turn off the external field, all the residual charge on the conducting halves will run to the flat part where we cut them. Hence, the force will be the same as that due to two conducting plates in the shape of disks. In other words, the problem is now to find the force of attraction between two oppositely charged disks of radius a. Ignoring the fringe electric fields, we have an electric field between the disks given by σ/ε_0 with σ uniform. Initially, we had a charge of

$$q = \int_0^{\pi/2} 3\varepsilon_0 E_0 \cos\theta \sin\theta \, \mathrm{d}\theta \cdot 2\pi a^2$$
$$= 3\pi\varepsilon_0 E_0 a^2.$$

So now our $\sigma = q/\pi a^2$. Hence, the force is

$$\frac{1}{2\varepsilon_0} \int \left(\frac{q}{\pi a^2}\right)^2 \mathrm{d}a = \frac{1}{2\varepsilon_0} \frac{q^2}{\pi a^2} = \frac{9\pi\varepsilon_0 {E_0}^2 a^2}{2},$$

so the force after we turn off the external field is exactly *twice* that of the prior one.

Exercise 2. Recall the image solution to a point charge outside a grounded conducting sphere: a charge $q' = -qa/b$ at a distance $b' = a^2/b$ from the center of the sphere, where the charge q is at a distance b from the center of the sphere of radius a. On the homework, you found (in Griffiths 3.9) the solution to this configuration where the sphere is a *neutral* conducting sphere.

a) Find the energy of the configuration in Griffiths 3.9.

b) Find the image solution to a point dipole with dipole moment $\mathbf{p}$ placed at a distance b from the center of a neutral conducting sphere of radius a in the two orientations:

   i) The dipole points in the direction towards the center of the sphere.

   ii) The dipole is perpendicular to the direction in (a).

a) Notice that if we place a point charge $q'' = 4\pi\varepsilon_0 a V_0$ at the center of the sphere, the sphere will have potential exactly V_0. Since we want the sphere to be neutral, we better take $q' = -q''$. Hence, we are in the following situation: There is a charge q, a charge q' at a distance $b - a^2/b$ from q, and a charge $q'' = -q'$ at a distance a^2/b from q' and b from q. All the charges sit on one line.

Calculating the force on q is now straightforward. Its magnitude is (dropping the $1/4\pi\varepsilon_0$)

$$q^2 \frac{a}{b}\left(\frac{1}{(b - a^2/b)^2} - \frac{1}{b^2}\right) = q^2 \frac{a}{b} \cdot \frac{2a^2 - a^4/b^2}{(b - a^2/b)^2 b^2} = \frac{q^2 a^3}{b^3} \frac{2b^2 - a^2}{(b^2 - a^2)^2}.$$

This force is an attractive force, since the charge is attracted to the neutral sphere.

To calculate the energy, just integrate the force on q from ∞ to b, which will be the work required to bring in q from infinity. Note that b is the variable here, so we should integrate

$$\int_\infty^b \frac{q^2 a^3}{x^3} \frac{2x^2 - a^2}{(x^2 - a^2)^2} dx.$$

Let's break this up into two integrals. To integrate something of the form

$$\int \frac{1}{x(x^2 - a^2)^2} dx,$$

we should use partial fractions. This is easiest with a substitution $u = x^2$, so that this integral becomes

$$\int \frac{du}{2u(u-a^2)^2}.$$

Then just do partial fractions:

$$\frac{1}{u(u-a^2)^2} = \frac{A}{u} + \frac{B}{u-a^2} + \frac{C}{(u-a^2)^2}.$$

This has the solution

$$\begin{aligned} A =& \frac{1}{a^4} \\ B =& -\frac{1}{a^4} \\ C =& \frac{1}{a^2}. \end{aligned}$$

Thus, the solution to this integral is (I'm not adding an integration constant, because we're doing a definite integral)

$$\frac{1}{2a^4}(2\ln(x) - \ln(x^2-a^2)) - \frac{1}{2a^2(x^2-a^2)}.$$

We do the same for the second integral, which in terms of u looks like

$$\int \frac{du}{u^2(u-a^2)^2}.$$

This has the partial fraction decomposition

$$\frac{1}{u^2(u^2-a^2)^2} = \frac{A}{u} + \frac{B}{u^2} + \frac{C}{u-a^2} + \frac{D}{(u-a^2)^2}.$$

This has the solution

$$\begin{aligned} A =& \frac{2}{a^6} \\ B =& \frac{1}{a^4} \\ C =& -\frac{2}{a^6} \\ D =& \frac{1}{a^4}. \end{aligned}$$

Thus, this integral is

$$\frac{1}{a^6}(2\ln(x) - \ln(x^2-a^2) - \frac{2x^2-a^2}{2a^4x^2(x^2-a^2)}.$$

Now, let's go back to our original integral. We see that the log terms cancel (they better, since they diverge at ∞), so we can compute the definite integral to be

$$q^2a^3\left(-\frac{1}{a^2(b^2-a^2)}-\frac{2b^2-a^2}{2a^2b^2(b^2-a^2)}\right).$$

In discussion, I wanted you to compute the energy of the configuration of *the charges*. Although this was more useful preparation for the exam than solving an integral, this is *not* the energy of the actual physical system! You can see this by going through the calculation, where we get a different answer: We can imagine first bringing in the first charge, q, then the second, q', and finally the third, q''. The first charge costs no work, the second costs

$$\frac{q^2a}{b}\int_{\infty}^{b-a^2/b}\frac{1}{x^2}\,\mathrm{d}x=-\frac{q^2a}{b^2-a^2}.$$

The third costs

$$\left(\frac{qa}{b}\right)^2\int_{\infty}^{a^2/b}\frac{1}{x^2}\,\mathrm{d}x-\frac{q^2a}{b}\int_{\infty}^{b}\frac{1}{x^2}\,\mathrm{d}x=\frac{q^2}{b}-\frac{q^2a}{b^2}.$$

Adding these together, we have

$$q^2\left(\frac{b-a}{b^2}-\frac{a}{b^2-a^2}\right),$$

and you can add in the factor of $1/4\pi\varepsilon_0$.

b) Case (ii). Consider the dipole as two charges $q,-q$ which are at a finite separation d from each other (we will later take $d\to 0$ and $q\to\infty$). Using part (a), we can find the image configurations for both charges: We have a charge $qa/(b-d/2)$ at a distance $a^2/(b-d/2)$ from the center of the sphere and a charge $-qa/(b-d/2)$ at the center of the sphere due to the negative charge $-q$, which is at a distance $b-d/2$ from the center of the sphere. Similarly, we have a charge $-qa/(b+d/2)$ at a distance $a^2/(b+d/2)$ from the center of the sphere and a charge $qa/(b+d/2)$ at the center of the sphere. Altogether, at the center of the sphere we have a charge $dqa/(b^2-d^2/4)$, and we have the two charges near the image location. Now, $qd=p$, and we will keep this fixed as we take $d\to 0$ and $q\to\infty$. Hence, the charge at the center will have magnitude pa/b^2. What about at the image location? The charges there are separated by a distance

$$\frac{a^2}{b-d/2}-\frac{a^2}{b+d/2}=\frac{a^2d}{b^2-d^2/4}.$$

As we take the limit, they will induce a *dipole* at the image location with dipole moment magnitude

$$\frac{pa^3}{b^3}.$$

On the other hand, the charges at the image location have difference in charge

$$\frac{qa}{b-d/2} - \frac{qa}{b+d/2} = \frac{qad}{b^2-d^2/4},$$

and this will go to

$$pa/b^2$$

as we pass to the limit. Hence, at the image location, there is both a dipole with moment $\mathbf{p} = \mathbf{p}a^3/b^3$ *and* a charge of magnitdude pa/b^2. Also, we have the charge at the center of the sphere, altogether two charges and a dipole form the image to the dipole in case 1.

Case (ii). We use the same method as in case 1, but this time the dipole is vertically situated. So the charges are at *equal* distances from the center of the sphere. In this case, the image point charges all cancel out (remember, it was the fact that they were at slightly different distances from the center of the sphere that gave us extra point charges). Hence, all we're left with is a single dipole at the image location with $\mathbf{p} = \mathbf{p}a^3/b^3$.

Exercise 3. The region between two parallel infinite conducting plates at $x = 0$ and $x = L$ is filled with charge of charge density $\rho = \rho_0 \sin(\pi x/L)$. Find the potential and electric field between the plates.

We need to solve Poisson's equation $\nabla^2 V = \rho$ in the region between the plates. Since the situation is independent of y and z, we can just take this to be an ODE in one variable:

$$\frac{\mathrm{d}^2 V}{\mathrm{d}x^2} = \rho_0 \sin(\pi x/L).$$

This has the solution (subject to the boundary conditions $V(0) = 0$ and $V(L) = 0$)

$$V(x) = -\frac{\rho_0 L^2}{\pi^2} \sin(\pi x/L).$$

The solution is obtained by noting that, in this case, Laplace's equation has solution $V \equiv 0$, so that we can solve Poisson's equation directly to obtain a solution.

Exercise 4. ***Griffiths 3.55.*** a) A long metal pipe of square cross-section (side a) is grounded on three sides, while the fourth (insulated from the rest) is maintained at constant potential V_0. Show that the net charge per unit length on the side opposite V_0 is

$$\lambda = -\frac{\varepsilon_0 V_0}{\pi} \ln 2.$$

b) A long metal pipe of circular cross-section of radius R is divided lengthwise into four equal sections, three of them grounded and the fourth maintained at constant potential V_0. Show that the net charge per unit length on the section opposite V_0 is the same as in (a).

a) Arrange the pipe so that the situation is independent of z and so that the boundary conditions give $V(0, y) = V_0$, $V(a, y) = 0$, $V(x, 0) = 0$, $V(x, a) = V_0$. We can then solve Laplace's equation in x and y only, so that writing $V(x, y) = X(x)Y(y)$, we have

$$X = A\sin(\alpha x) + B\cos(\alpha x)$$
$$Y = Ce^{\alpha y} + De^{-\alpha y}.$$

Imposing the boundary conditions at $x = 0$, $x = a$, and $y = 0$, we find that the ones for x give $B = 0$ and $\alpha = n\pi/a$, where $n \in \mathbb{N}$, while those for Y give that $C = -D$. It follows that we can write

$$Y = C\sinh(\frac{n\pi}{a}y)$$
$$X = A\sin\left(\frac{n\pi}{a}x\right)$$

It follows that the full solution is given by

$$V(x, y) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{a}x\right)\sinh\left(\frac{n\pi}{a}y\right).$$

The only remaining boundary condition is that at $y = a$. It gives

$$V_0 = \sum A_n \sin\left(\frac{n\pi}{a}x\right)\sinh(n\pi).$$

Multiplying both sides by $\sin(m\pi x/a)$, where $m \in \mathbb{N}$, and integrating, we get that the sum on the right hand side will collapse by orthogonality of the sine functions. Thus, we'll have

$$-V_0\frac{a}{m\pi}\cos\left(\frac{m\pi}{a}x\right)\Big|_0^a = \frac{a}{2}A_m\sinh(m\pi).$$

Solving for A_m, we have

$$A_m = -\frac{2V_0}{m\pi\sinh(m\pi)}\left[(-1)^m - 1\right].$$

Now, the term in brackets is 0 if m is even and -2 if m is odd; hence,

$$A_m = \begin{cases} 0, & m \text{ even} \\ \frac{4V_0}{m\pi\sinh(m\pi)}, & m \text{ odd} \end{cases}.$$

Finally, we can write the full solution

$$V(x, y) = \sum_{k=0}^{\infty} \frac{4V_0}{(2k+1)\pi\sinh[(2k+1)\pi]}\sin\left(\frac{(2k+1)\pi}{a}x\right)\sinh\left(\frac{(2k+1)\pi}{a}y\right).$$

To find the surface charge on the side opposite V_0, we need to calculate

$$\sigma = -\varepsilon_0 \frac{\partial V}{\partial y}\bigg|_{y=0}.$$

This will give

$$\sigma = \sum_{k=0}^{\infty} \frac{4\varepsilon_0 V_0}{a \sinh[(2k+1)\pi]} \sin\left(\frac{(2k+1)\pi}{a} x\right).$$

Now, to calculate λ, we just need to integrate this over the x-direction. Thus,

$$\begin{aligned}\lambda &= \int_0^a \sigma \,\mathrm{d}x \\ &= \sum_{k=0}^{\infty} \frac{8\varepsilon_0 V_0}{(2k+1)\pi \sinh[(2k+1)\pi]},\end{aligned}$$

since

$$\int_0^a \sin\left(\frac{(2k+1)\pi}{a} x\right) \mathrm{d}x = \frac{a}{(2k+1)\pi}\left[\cos((2k+1)\pi) - 1\right] = -\frac{2a}{(2k+1)\pi}.$$

Now, I wasn't able to figure out how to sum this analytically; however, Mathematica gave that

$$\sum_{k=0}^{\infty} \frac{8}{(2k+1)\sinh[(2k+1)\pi]} \approx 0.0866434$$

and

$$\ln(2)/8 \approx 0.0866434,$$

so they match exactly.

b) We need to solve essentially the same problem as in (a), but this time in cylindrical coordinates. Set up the problem so that the boundary conditions are

$$\begin{aligned} s = R,\ \varphi \in (-\pi/4, \pi/4) &\implies V = V_0 \\ s = R,\ \varphi \notin (-\pi/4, \pi/4) &\implies V = 0, \end{aligned}$$

where we take $\varphi \in [-\pi, \pi]$. The situation is independent of z, so we can solve Laplace's equation in cylindrical coordinates:

$$\frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial V}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 V}{\partial \varphi^2} = 0.$$

Writing $V = S(s)\Phi(\varphi)$, we have

$$s^2\frac{S''}{S} + s\frac{S'}{S} + \frac{\Phi''}{\Phi} = 0.$$

As in the case of cartesian coordinates, the sum of the first two terms is a constant, α^2, say, and the second term is likewise equal to $-\alpha^2$. Hence, we have two differential equations, and we can immediately write down the solution to the one for φ. It is

$$\Phi = A\sin(\alpha\varphi) + B\cos(\alpha\varphi).$$

Now, recall that $\Phi(\varphi) = \Phi(\varphi + 2\pi)$, so that α must be an integer, say $n \in \mathbb{Z}$. We can plug this in to the differential equation for s:

$$s^2 S'' + sS' = n^2 S.$$

It's easy to see that this has solutions

$$s^{\pm n}$$

for $n \neq 0$. Now, if $n = 0$, then we end up with the equation

$$sS'' = -S'.$$

We can solve this by writing $u = S'$, so that we instead need to solve

$$u' = -\frac{1}{s}u,$$

which is separable. Its solution is

$$\ln u = -\ln s + C,$$

so

$$S' = \frac{C}{s},$$

where we've redefined C. Solving this for S, we have

$$S = C\ln(s) + D.$$

Let's put it all together. For $V = \sum_n S_n \Phi_n$, our solutions are given by

$$V(s,\varphi) = C_0 \ln(s) + D_0 + \sum_{n=1}^{\infty} \left(C_n s^n + D_n s^{-n}\right)\left(A_n \sin(n\varphi) + B_n \cos(n\varphi)\right).$$

Now, we can finally use our boundary conditions. First of all, the solutions with s^{-n} are not physical, since they blow up at $s = 0$. Thus, $D_n = 0$. Similarly, $\ln(s)$ blows up at $s = 0$, so $C_0 = 0$ as well. Lastly, sin is antisymmetric in φ, while our boundary conditions are symmetric about the origin in $\varphi \in [-\pi, \pi]$; thus, $A_n = 0$. We are thus left with

$$V(s,\varphi) = A + \sum_{n=1}^{\infty} A_n s^n \cos(n\varphi),$$

where we've again redefined the constants. When $s = R$, we have

$$V(R, \varphi) = A + \sum_{n=1}^{\infty} A_n R^n \cos(n\varphi).$$

First, multiply both sides by $\cos(m\varphi)$, where $m \neq 0$, and integrate from $-\pi$ to π. We can split up the integral on the left hand side into two: one where $V(R, \varphi)$ is 0 and one where $V(R, \varphi) = V_0$. Hence, we have

$$\int_{-\pi/4}^{\pi/4} V_0 \cos(m\varphi)\, \mathrm{d}\varphi = A_m R^m \pi,$$

so that

$$A_m = \frac{2V_0}{mR^m \pi} \sin\left(\frac{m\pi}{4}\right)$$

when $m \neq 0$. In the case $m = 0$, $\cos(m\varphi) = 1$, so just integrate both sides from 0 to 2π to kill the sum and obtain

$$V_0 \frac{\pi}{2} = 2\pi A.$$

Thus,

$$A = \frac{V_0}{4}.$$

We have our answer:

$$V(s, \varphi) = \frac{V_0}{4} + \sum_{n=1}^{\infty} \frac{2V_0 \sin(n\pi/4)}{nR^n \pi} s^n \cos(n\varphi).$$

To compute the surface charge on the side opposite V_0, we take

$$\begin{aligned}\sigma =& -\varepsilon_0 \frac{\partial V}{\partial s}\bigg|_{s=R,\ \varphi\in[3\pi/4,\pi]\cup[-\pi,-3\pi/4]} \\ =& -\sum_{n=1}^{\infty} \frac{2\varepsilon_0 V_0}{R\pi} \sin(n\pi/4) \cos(n\varphi).\end{aligned}$$

To compute λ, we just integrate:

$$\begin{aligned}\lambda =& \int_{-\pi}^{-3\pi/4} \sigma\, \mathrm{d}\varphi + \int_{3\pi/4}^{\pi} \sigma\, \mathrm{d}\varphi \\ =& 2\int_{3\pi/4}^{\pi} \sigma\, \mathrm{d}\varphi \\ =& \sum_{n=1}^{\infty} \frac{4\varepsilon_0 V_0}{nR\pi} \sin(n\pi/4) \sin(3n\pi/4),\end{aligned}$$

since

$$\int_{3\pi/4}^{\pi} \sin(n\varphi)\, \mathrm{d}\varphi = -\sin(3n\pi/4).$$

Unlike the sum in (a), we can evaluate this sum analytically. Let's do so, but before then, we can check that we got the right answer with Mathematica. Indeed, we have that the sum is approximately equal to

$$\sum_{n=1}^{\infty} \frac{\sin(n\pi/4)\sin(3n\pi/4)}{n} \approx 0.173287 \approx \ln(2)/4.$$

OK, let's try and tackle this thing. We want to sum

$$\sum_{n=1}^{\infty} \frac{\sin(n\pi/4)\sin(3n\pi/4)}{n}.$$

Use the product identity to get that this is the same as

$$\sum \frac{\cos(n\pi/2) - \cos(n\pi)}{2n}.$$

Now,

$$\cos(n\pi/2) = \begin{cases} (-1)^{n/2}, & n \text{ even} \\ 0, & n \text{ odd} \end{cases},$$

and

$$\cos(n\pi) = (-1)^n.$$

Hence,

$$\sum \frac{\cos(n\pi/2) - \cos(n\pi)}{2n} = \frac{1}{2}\sum_{k=1}^{\infty}\left(\frac{(-1)^k}{2k} - \frac{(-1)^k}{k}\right),$$

where we've changed variables $n = 2k$ in the first term and set $n = k$ in the second. Simplifying, we get

$$\frac{1}{2}\sum_{k=1}^{\infty}(-1)^k\left(\frac{1}{2k} - \frac{1}{k}\right) = -\frac{1}{4}\sum_{k=1}^{\infty}(-1)^k\frac{1}{k}.$$

Phew! We finally have a nice sum. Notice that by the alternating series test, the sum

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{k}$$

is convergent! It must be equal to $-\ln(2)$ to give us the right answer. Recall (or look up) the power series for $\ln(1+x)$:

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^n \frac{x^n}{n},$$

and notice that for $x = 1$ this series exactly matches negative the one we have obtained. Hence, it must sum to $-\ln(2)$. So we're done!

Remark. This problem is *much* harder than anything you'd be expected to do on the exam, so you will definitely be prepared if you solved it!

Exercise 5. ***Griffiths 3.28.*** A charge is distributed with uniform linear charge density λ over the circumference of a circle of radius R which lies in the (x, y)-plane with center at the origin.

a) Find the potential $V(z)$ on the z-axis.

b) Find the first three terms in the multipole expansion for $V(r, \theta)$.

a) The distance from the circle to a point on the z-axis is

$$\sqrt{R^2 + z^2},$$

so the potential is

$$\begin{aligned} V &= \int_0^{2\pi} \frac{\lambda R \, \mathrm{d}\varphi}{\sqrt{R^2 + z^2}} \\ &= \frac{2\pi\lambda R}{\sqrt{R^2 + z^2}}. \end{aligned}$$

b) The first term is

$$\frac{1}{r} \int_0^{2\pi} \lambda R \, \mathrm{d}\varphi = \frac{2\pi\lambda R}{r}.$$

The second includes an angle α with $\cos\alpha = \hat{r} \cdot \hat{r}'$. We can write

$$\begin{aligned} \hat{r} &= (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta) \\ \hat{r}' &= (\cos\varphi', \sin\varphi', 0), \end{aligned}$$

so that

$$\cos\alpha = \sin\theta(\cos\varphi\cos\varphi' + \sin\varphi\sin\varphi').$$

Now, we can do the integral with $r' = R$

$$\begin{aligned} V_{\text{dip}}(\mathbf{r}) &= \frac{1}{r^2} \int_0^{2\pi} R\sin\theta(\cos\varphi\cos\varphi' + \sin\varphi\sin\varphi')\lambda R \, \mathrm{d}\varphi' \\ &= 0. \end{aligned}$$

Similarly, we can do the quadrupole integral

$$\begin{aligned}
V_{\text{quad}}(\mathbf{r}) =& \frac{1}{r^3}\int_0^{2\pi} R^2\left(\frac{3}{2}\sin^2\theta(\cos\varphi\cos\varphi' + \sin\varphi\sin\varphi')^2 - \frac{1}{2}\right)\lambda R\,\mathrm{d}\varphi' \\
=& \frac{\lambda R^3}{2r^3}\int_0^{2\pi}\left(\frac{3}{2}\sin^2\theta(\cos^2\varphi\cos^2\varphi' + \sin^2\varphi\sin^2\varphi') - \frac{1}{2}\right)\mathrm{d}\varphi' \\
=& \frac{\lambda R^3}{4r^3}\left(3\pi\sin^2\theta - 2\pi\right).
\end{aligned}$$

Exercise 6. Six equal by absolute value charges are placed at the vertices of a regular hexagon. The signs of any two neighboring charges are opposite. What kind of multipole does the following system form? By what power law does the potential decay at large distances r from the center of the hexagon?

The total charge of the configuration is 0, so there is no monopole term. Similarly, the dipole moment has x-coordinate

$$\begin{aligned}
p_x =& \int x'\rho(\mathbf{r}')\,\mathrm{d}V' \\
=& aq - aq + 2aq\cdot\frac{1}{2} - 2aq\cdot\frac{1}{2} = 0,
\end{aligned}$$

where a is the distance from the origin to any of the charges and the charges have magnitude q. You can compute similarly for the y-coordinate that it's 0.

What about the quadrupole term? You could compute this by explicitly working through the definitions, but there is an easier way. Since $P_n(\cos\alpha)$ is odd for n odd and even for n even, we have that $\sum_i q_i P_2(\cos\alpha_i) = 0$ for the hexagon, since it has odd symmetry (the regular hexagon itself has even symmetry, but the negative charges make it have odd symmetry in this example).

What about the octopole term? Since $P_3(\cos\alpha) = \frac{5\cos^3\alpha - 3\cos\alpha}{2}$, we need to only check that this does not vanish to see if we get an octopole. Let's place the hexagon in the (x, y)-plane, with two charges on the x-axis, and let's try to compute the octopole potential on the x-axis. Just looking at $\sum q_i\cos^3\alpha_i$, we have

$$\begin{aligned}
& q\left(1 - (-1) - \cos^3(\pi/3) + \cos^3(\pi + \pi/3) + \cos^3(2\pi/3) - \cos^3(\pi + 2\pi/3)\right) = \\
=& q\left(2 - 2\cos^3(\pi/3) + 2\cos^3(2\pi/3)\right).
\end{aligned}$$

Similarly, $\sum q_i\cos\alpha_i$ gives

$$q\left(2 - 2\cos(\pi/3) + 2\cos(2\pi/3)\right),$$

and it's clear that their weighted sum $\sum q_i P_3(\cos\alpha_i) \neq 0$. Therefore, this system has an octopole moment and hence has potential that dies off as $1/r^4$ at large distances.

Exercise 7. The center of a metal sphere of radius a lies on the flat boundary between two dielectric regions of permittivities ε_1 and ε_2. At a distance b from the center of the sphere in the region with permittivity ε_1 is placed a point charge q.

a) Find the potential of the sphere if it is insulated and uncharged.
Hint: If you find the total charge which is induced in the dielectrics, then you can use your solution to Griffiths 3.9 from Homework 3 to determine the potential of the sphere. You should get

$$V = \frac{q}{2\pi b(\varepsilon_1 + \varepsilon_2)}.$$

b) Find the charge induced on the sphere if it is grounded.

a) Before we start solving the problem, let's make some general considerations. Recall from Exercise 2/Griffiths 3.9 that if we're in the same situation but in vacuum, then the sphere will be at potential

$$V = \frac{q}{2\pi b\varepsilon_0},$$

since we place a charge $q'' = qa/b$ at the center of the sphere to account for the fact that it isn't grounded. We just need to figure out how this potential will change if we are now in the situation in question. The idea is the same: We should look for an image charge at the center which accounts for the *effective* point charge which is induced by the dielectric (plus the actual point charge). So we need to figure out 1) the charge that accumulates in ε_1 due to the point charge and 2) the charge that accumulates in ε_2. Then, we could just plug that in to the above formula, and we're done.

Since $\rho_b = -\nabla \cdot \mathbf{P}$,

$$\begin{aligned}\rho_b &= -\frac{\varepsilon_r - 1}{\varepsilon_r}\nabla \cdot \mathbf{D} \\ &= -\frac{\varepsilon_r - 1}{\varepsilon_r}\rho_f.\end{aligned}$$

In the region ε_1, we thus have that the charge which accumulates around q is

$$q_b = -\frac{\varepsilon_r - 1}{\varepsilon_r}q$$

The total charge there is hence

$$q + q_b = \frac{q}{\varepsilon_{1r}},$$

where $\varepsilon_{1r} = \frac{\varepsilon_1}{\varepsilon_0}$. There is also bound surface charge which will accumulate at the interface between the regions (and on both sides of the interface!). We can now follow Example 4.8 in Griffiths. The bound surface charge is

$$\begin{aligned}\sigma_b =& \mathbf{P} \cdot \hat{n} \\ =& (\varepsilon_1 - \varepsilon_0)E_z\end{aligned}$$

if we suppose that the boundary region lies in the (x, y)-plane. The total bound surface charge in region ε_1 is then

$$\sigma_{b1} = (\varepsilon_1 - \varepsilon_0)\left(\frac{qb}{4\pi\varepsilon_1(r^2 + b^2)^{3/2}} - \frac{\sigma_{b1}}{2\varepsilon_0} - \frac{\sigma_{b2}}{\varepsilon_0}\right),$$

where we suppose that the region with ε_1 is above the z-axis. Then

$$\sigma_{b2} = (\varepsilon_2 - \varepsilon_0)\left(-\frac{qb}{4\pi\varepsilon_1(r^2+b^2)^{3/2}} - \frac{\sigma_{b1}}{2\varepsilon_0} - \frac{\sigma_{b2}}{\varepsilon_0}\right).$$

We can solve these two equations for σ_{bi} to get

$$\begin{aligned}\sigma_{b1} =& \frac{qb\varepsilon_{2r}(\varepsilon_{1r}-1)}{2\pi(r^2+b^2)^{3/2}\varepsilon_{1r}(\varepsilon_{1r}+\varepsilon_{2r})} \\ \sigma_{b2} =& -\frac{qb(\varepsilon_{2r}-1)}{2\pi(r^2+b^2)^{3/2}(\varepsilon_{1r}+\varepsilon_{2r})}.\end{aligned}$$

Now, add these up, and integrate over the (x, y)-plane (you can just use that the integral of $b(r^2 + b^2)^{-3/2}$ is 2π from the example in Griffiths) to get that the total bound charge is

$$q_{\text{interface}} = \frac{\varepsilon_1 - \varepsilon_2}{\varepsilon_1 + \varepsilon_2}\frac{q}{\varepsilon_{1r}}.$$

Phew! Now, we can add this up with the contribution due to the charge at the location of q, $\frac{q}{\varepsilon_{1r}}$, to get

$$\frac{2q}{\varepsilon_{1r} + \varepsilon_{2r}}.$$

Indeed, we can think of the surface charge induced on the interface of the dielectrics as the charge induced on a conductor in the image problem with a charge above a conducting plane. We know this is always equal and opposite to the image charge, so we should just add the contribution to determine the image to this at the location of q. Now, we can use the result of Griffiths 3.9 from the homework with a single charge

$$q_{\text{tot}} = \frac{2q}{\varepsilon_{1r} + \varepsilon_{2r}}$$

to find that the potential will be

$$V = \frac{q}{2\pi b(\varepsilon_1 + \varepsilon_2)}.$$

b) We can again use Griffiths 3.9. If the sphere is grounded, the charge induced on the sphere will be the same as the image charge. This time, we don't put any charge in the middle of the sphere, so the image charge is just $-q_{\text{tot}}a/b$.

Exercise 8. A half-infinite dipole string with linear dipole moment density $\mathbf{p}_l = p_l\hat{x}$ is placed along the negative z-axis.

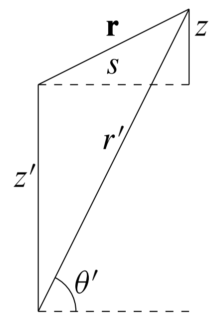
a) The potential due to a dipole *at the origin* is given by

$$V(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\frac{\mathbf{p}\cdot\hat{r}}{r^2}.$$

Generalize this to find the potential $V(x, y, z)$ due to the string. This is probably easier if you use cylindrical coordinates.

b) Investigate the behavior of the potential from (a) as you approach the z-axis in the regions $z < 0$ and $z > 0$.

a) In general, r will be the distance from a point on the negative z-axis to a point where we'd like to calculate the potential. Similarly, $\mathbf{p} \cdot \hat{r}$ will be $\cos(\theta') p_l$, where θ' is the angle between a vector $\mathbf{r}'$ from a point on the negative z-axis to the point where we'd like to calculate the potential and the x-axis (since $\mathbf{p}$ is along $\hat{x}$). To compute the total potential, we'd then integrate along the negative z-axis. Let's calculate each of these in cylindrical coordinates, but it helps to first draw a picture:



Notice that from the picture we can read off that $\cos\theta' = \frac{s}{r'}$ and $r'^2 = s^2 + (z + z')^2$. Hence, we have

$$
\begin{aligned}
V(\mathbf{r}) &= \int_0^{-\infty} \frac{p_l s}{r'^3}\, \mathrm{d}z' \\
&= \int_0^{-\infty} \frac{p_l s}{\left[s^2 + (z + z')^2\right]^{3/2}}\, \mathrm{d}z' \\
&= \left.\frac{z' + z}{s\sqrt{s^2 + (z + z')^2}}\right|_0^{-\infty} \\
&= -\frac{1}{s} - \frac{z}{s\sqrt{s^2 + z^2}}.
\end{aligned}
$$

To convert back to cartesian coordinates, we need only write $s = \sqrt{x^2 + y^2}$, so that

$$
V(\mathbf{r}) = -\frac{1}{\sqrt{x^2 + y^2}} - \frac{z}{\sqrt{(x^2 + y^2)(x^2 + y^2 + z^2)}}.
$$

b) Now, let's investigate what happens as we approach the z-axis. This is easier to do with the cylindrical coordinate version of our result above. Notice that the function we obtain has a pole of order 1 in s (for any value of z). Now, if $z \leq 0$, then our function approaches ∞ as $s \to 0^-$ and it approaches $-\infty$ as $s \to 0^+$. Hence, there is an infinite discontinuity at the z-axis, when $s = 0$. On the other hand, if $z > 0$, then our function approaches 0 as $s \to 0$ from either the positive direction or the negative one. Hence, the function is *continous at $s = 0$ in this case*! This means that the infinite line of dipoles only causes a divergence in our potential when we get near it—if we're in the region $z > 0$ there is no divergence at all.

Remark. What we are witnessing in this exercise is a breakdown of the space in which we are doing physics. Usually, we are able to define a *continuous* potential function on all of $\mathbb{R}^3$; however, it seems that now the negative z-axis is “not allowed.” This means that the topology is no longer $\mathbb{R}^3$, but $\mathbb{R}^3 \setminus \mathbb{R}^+$, or something like that. Later on, we’ll talk about solenoids: These have a magnetic field inside but none outside. In the Aharanov-Bohm effect, an electron will “feel the effect” of the magnetic field inside a solenoid via the vector potential (i.e. a vector $\mathbf{A}$ such that $\nabla \times \mathbf{A} = \mathbf{B}$, where $\mathbf{B}$ is the magnetic field), which is *not* zero outside the solenoid. This is because the electron somehow knows that the topology of space changed from $\mathbb{R}^3$ to $\mathbb{R}^3$ with a hole in it (the hole is the solenoid). This has deep connections to magnetic monopoles and other interesting phenomena, and this exercise is a toy example of this.