Final Review Problems Solutions

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Exercise 1. A charge is distributed with linear charge density λ over the circumference of a circle of radius R which lies in the (x, y)-plane with center at the origin. Find the potential $V(z)$ on the z-axis in the following cases.

a) λ is uniform.

b) $\lambda = C\sin(n\theta)$, where $n \in \mathbb{N}$, C is a constant, and θ is the polar angle.

c) $\lambda = C\theta$.

a) Since λ is constant and $\mathscr{r} = \sqrt{z^2 + R^2}$ for a point z on the z-axis, we have

$$\begin{aligned} V(z) =& \frac{1}{4\pi\varepsilon_0}\int \frac{\lambda}{\mathscr{r}} R\,\mathrm{d}\varphi \\ =& \frac{R\lambda}{2\varepsilon_0\sqrt{z^2 + R^2}}. \end{aligned}$$

b) Now, we need to actually do the integral over φ. We get

$$-\frac{RC}{4\pi\varepsilon_0\sqrt{z^2 + R^2}} \cdot \frac{1}{n}\cos(n\theta)\bigg|_0^{2\pi} = 0.$$

c) We now get

$$V(z) = \frac{\pi RC}{2\varepsilon_0\sqrt{z^2 + R^2}}.$$

Exercise 2. ***Griffiths 5.13.*** Suppose you have two infinite, parallel line charges λ a distance d apart, which are moving at a constant speed v. How great would v have to be for the magnetic attraction to balance the electrical repulsion? Calculate the number, and comment on the result.

Recall that the magnetic field due to an infinite wire at distance d is

$$\mu_0 I/2\pi d.$$

The force is then $\mathbf{F} = q\mathbf{v} \times \mathbf{B}$, so

$$F = \lambda v \frac{\mu_0 \lambda v}{2\pi d}.$$

This should be equal to the force due to the electric field which is qE. The electric field due to the wire can be determined by Gauss' law to be

$$E = \frac{\lambda}{2\pi d \varepsilon_0}.$$

Setting these equal, we find

$$\begin{aligned} v =& \frac{1}{\sqrt{\mu_0 \varepsilon_0}} \\ =& c. \end{aligned}$$

So we find that these forces will only be equal when v is the speed of light!

Exercise 3. ***Griffiths 6.15.*** If $\mathbf{J}_f = \mathbf{0}$ everywhere, the curl of $\mathbf{H}$ vanishes, so we can express $\mathbf{H}$ as the gradient of a scalar potential W,

$$\mathbf{H} = -\boldsymbol{\nabla} W.$$

Thus,

$$\nabla^2 W = \boldsymbol{\nabla} \cdot \mathbf{M},$$

so W obeys Poisson's equation with $\boldsymbol{\nabla} \cdot \mathbf{M}$ as the "source." As an example, find the field inside a uniformly magnetized sphere by separation of variables.

For a uniformly magnetized sphere of radius R, the magnetization is

$$\mathbf{M} = \begin{cases} M\hat{z}, & r < R \\ \mathbf{0}, & r > R \end{cases}.$$

We can use the boundary condition

$$H^{\perp}_{>R} - H^{\perp}_{<R} = M^{\perp}_{<R} - M^{\perp}_{>R}$$

to get

$$-(\nabla W_{>R})_r + (\nabla W_{<R})_r = M \cos\theta.$$

Also, we have continuity at $r = R$:

$$W_{<R}(R) = W_{>R}(R).$$

Using the general solution for Laplace's equation in spherical coordinates, we have that

$$\sum_{\ell=0}^{\infty} \ell A_\ell r^{\ell-1} P_\ell(\cos\theta) + \sum_{\ell=0}^{\infty} (\ell+1) B_\ell r^{-\ell-2} P_\ell(\cos\theta) = M\cos\theta,$$

and

$$\sum A_\ell R^{2\ell+1} P_\ell(\cos\theta) = \sum B_\ell P_\ell(\cos\theta),$$

which implies that

$$B_\ell = A_\ell R^{2\ell+1},$$

since the Legendre polynomials are orthogonal. Now, we can use the first boundary condition to get that $A_\ell = 0$ for all $\ell \neq 1$, since the right hand side has $\cos\theta = P_1(\cos\theta)$. Thus,

$$A_1 + 2A_1 = M,$$

so

$$A_1 = \frac{M}{3}.$$

We can plug this in to get

$$W = \begin{cases} \frac{M}{3} z, & r < R \\ \frac{MR^3}{3r^3}\cos\theta, & r > R \end{cases}.$$

We can now find **H** inside the sphere, which will be given by

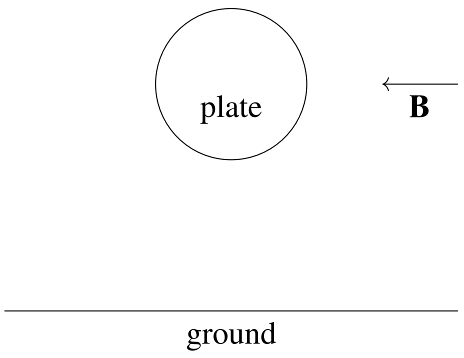
$$\begin{aligned} \mathbf{H} =& -\nabla W \\ =& -\frac{M}{3}\hat{z} \end{aligned}$$

So the field will be

$$\begin{aligned} \mathbf{B} =& \mu_0(\mathbf{H} + \mathbf{M}) \\ =& \mu_0 \frac{2M}{3}\hat{z}, \end{aligned}$$

which is the right answer!

Exercise 4. Find the acceleration a of a freely falling, circular, metal plate in a uniform magnetic field which is parallel to the surface of the ground. The plate is oriented with its normal vector (to the circular sides) perpendicular to the direction of the magnetic field and parallel to the ground. The radius of the plate is R and its thickness is $d \ll R$. Its mass is m and the strength of the magnetic field is B.



Let's think about what will physically happen as the plate falls. First of all, it has an acceleration due to gravity given by g. So the electrons inside the conductor will feel a force $q\mathbf{v}\times\mathbf{B}$ which will be toward one side of the plate. Now, notice that this force is the same as the electric force with an electric field $\mathbf{E}=\mathbf{v}\times\mathbf{B}$. Hence, the plate will now polarize as if it is in an external electric field $\mathbf{v}\times\mathbf{B}$. Now, the electric field inside the conductor will be

$$E=\frac{\sigma}{\varepsilon_0}.$$

Since $\sigma=\frac{q}{\pi R^2}$ is uniform, the induced charge on the opposite sides of the plate will then be

$$q=\pi R^2\varepsilon_0 vB.$$

But $\mathbf{v}$ is changing, so that we actually end up with a current which is flowing from one side of the plate to the other:

$$I=\pi R^2\varepsilon_0 aB.$$

This then gives another force on the plate which will point *up*. Thus, the acceleration of the plate will be *less than* g. This force will be

$$\begin{aligned}\mathbf{F}=&I\int \mathrm{d}\boldsymbol{\ell}\times\mathbf{B}\\=&IdB\\=&\pi R^2\varepsilon_0 aB^2d.\end{aligned}$$

Newton's law gives

$$ma=mg-\pi R^2\varepsilon_0 aB^2d,$$

and we can solve this for a to find

$$a=\frac{g}{1+\frac{\pi R^2\varepsilon_0 aB^2d}{m}}.$$

Exercise 5. A capacitor with circular plates is given an alternating voltage $V(t)=V_0\sin(\omega t)$. Find the magnetic field within the capacitor if $d\ll a\ll c/\omega$, where d is the distance between the plates and a is the radius of the plates.

The electric field between the plates is

$$E = V/d.$$

We can then use Maxwell's equation

$$\nabla \times \mathbf{B} = \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t}.$$

Integrating this equation around a disk of radius r placed between the plates and parallel to them, we find

$$\oint \mathbf{B} \cdot d\boldsymbol{\ell} = \frac{1}{c^2}\frac{d\Phi_E}{dt},$$

where Φ_E is the electric flux passing through the disk and we used Stokes' theorem on the LHS. Evaluating both sides, we find

$$B_\varphi \cdot 2\pi r = \frac{\omega}{c^2}\pi r^2 \frac{V_0 \sin(\omega t)}{d}.$$

Solving this for the φ-component of $\mathbf{B}$, we get

$$B_\varphi = \frac{\omega V_0 \sin(\omega t)}{2c^2 d},$$

where $\mathbf{E} = E\hat{z}$. We know $B_z = 0$ since $\mathbf{B}$ has to be perpendicular to $\mathbf{E}$, but we have not yet determined the r-component. This can be done by using Maxwell's equation

$$\nabla \cdot \mathbf{B} = 0.$$

Evaluating the divergence in cylindrical coordinates, we find

$$B_r + r\partial_r B_r = 0.$$

Thus,

$$\partial_r B_r = -\frac{1}{r}.$$

This has the solution

$$B_r = C/r$$

for some constant C. But this is unphysical at $r = 0$, so we find C must be 0.

Exercise 6. This Exercise is an add-on to Griffiths 7.44-46. Give the change of the self-inductance ΔL of a small circular wire loop if it is placed parallel to a superconducting plane and a distance h above it? Suppose that the radius of the loop is $a \ll h$, and recall that a superconductor can be thought of as a perfect conductor where $\mathbf{B} = \mathbf{0}$ inside.

Since $\mathbf{B} = \mathbf{0}$ just above the superconductor, this configuration will be satisfied with an image magnetic dipole pointing in the opposite direction. If the real dipole has dipole moment $m = I\pi a^2$, so does the image dipole. The magnetic field a distance $2h$ along the z-direction from the dipole is

$$-\frac{\mu_0}{16\pi h^3} m\hat{z}.$$

Since we're just interested in the *change* of the self-inductance, we don't need to worry about what L was to begin with. Using $\Phi = LI$, we find that we just need to compute the flux through the loop due to the image dipole and divide it by I. This is given by

$$\Delta L = -\frac{\pi a^4 \mu_0}{16h^3}.$$

Exercise 7. Find the energy of the configuration of an uncharged dielectric or metal object which under the effect of a fixed electric field $\mathbf{E}$ obtains a dipole moment $\mathbf{p}$. How will this energy change if the dipole moment $\mathbf{p}$ of the object does not depend on the external field $\mathbf{E}$?

Write $\mathbf{E} = -\nabla V_0$, where V_0 is some "external potential." Then the energy

$$W = \frac{1}{2}\int \rho_i V_0 d^3 r',$$

where ρ_i is the density of the induced charges in the material. Now, since we know that the object acquires only a dipole moment, we know that inside $\mathbf{E}$ is *uniform*. Thus, we can write

$$V_0 = -\mathbf{r}\cdot\mathbf{E} + C$$

for some constant C. This makes it easy to calculate the energy:

$$W = -\frac{1}{2}\mathbf{p}\cdot\mathbf{E}.$$

If the dipole moment doesn't depend on the external field, then suppose there is some ρ_1 which induces this dipole moment. Suppose the potential of these charges is V_1. Then the energy is now

$$W = \frac{1}{2}\int \rho_0 V_1 d^3 r' + \frac{1}{2}\int \rho_1 V_0 d^3 r',$$

where ρ_0 is the charge density of the external charges which create the field. Another way to see this is to take the total energy,

$$\frac{1}{2}\int (\rho_0 + \rho_1)(V_0 + V_1) d^3 r',$$

and subtract off the self-energy of ρ_0, ρ_1, which is

$$\frac{1}{2}\int \rho_1 V_1 d^3 r' + \frac{1}{2}\int \rho_0 V_0 d^3 r'.$$

Thus, our final answer is

$$W = \frac{1}{2}\int (\rho_0 V_1 + \rho_1 V_0) d^3r'.$$

We now claim that these two terms are actually *equal*, i.e.

$$\int \rho_0 V_1 d^3r' = \int \rho_1 V_0 d^3r'.$$

Since $\nabla^2 V_0 = -\rho_0/\varepsilon_0$ and similarly $\nabla^2 V_1 = -\rho_1/\varepsilon_0$, we can write

$$\int \rho_0 V_1 d^3r' = -\varepsilon_0 \int \nabla^2 V_0 V_1 d^3r' = -\varepsilon_0 \int V_0 \nabla^2 V_1 d^3r' = \int \rho_1 V_0 d^3r'.$$

by double integration by parts. Thus, we find

$$\begin{aligned} W &= \int \rho_1 V_0 d^3r' \\ &= -\mathbf{p}\cdot\mathbf{E}. \end{aligned}$$

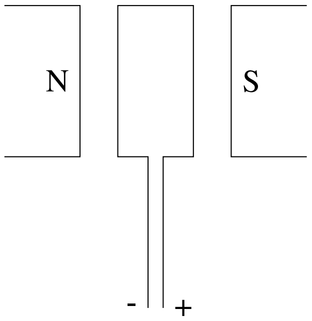
Exercise 8. A plasma can be modeled as a material (usually in gaseous form) in which most of the atoms are ionized. Explain why we should expect it to have an anisotropic dielectric tensor in the presence of an external uniform magnetic field (see Exercise 1 on the Week 15 Worksheet).

Let's think about what happens as we induce electric fields in the plasma in different directions, supposing that $\mathbf{B} = B\hat{z}$. If we pick an electric field along the z-direction, we should expect that there is some dielectric constant which controls this, and there won't be any "mixing," since $\mathbf{E}$ will be parallel to $\mathbf{B}$. On the other hand, as soon as we induce an electric field in the (x, y)-plane, the charges which begin to move along the electric field will also begin to rotate in the (x, y)-plane due to the $\mathbf{B}$ field. Thus, we should expect that there should be "mixing" between the x- and y-components of the dielectric tensor; in other words, we should not expect it to be diagonal. Now, we should explain why the eigenvalues should in general be different (this is what it means to be "anisotropic").

Taking the hint from the exercise on the Week 15 Worksheet that we should consider circular polarizations, let's think about what happens if we have charges rotating clockwise or counterclockwise about the $\mathbf{B}$ field. If the rotation vector points along $\hat{z}$, then the charges get pushed towards the center of the circle by the Lorentz force. This will cause them to repel each other, hence slow down. On the other hand, if the vector points in the opposite direction, then the particles will be pushed away from the center of the circle and won't feel as much of this electric repulsion. This should mean that these two "polarizations" inherit different dielectric constants, i.e. the dielectric tensor is anisotropic.

Exercise 9. Explain how an AC generator works.

The idea is to have a magnet with the north pole facing the south pole and a gap in between. In the gap, we place a coil of wire as shown in the figure below. Now, think about what happens as we rotate this loop about its symmetry axis (the vertical, or y-, axis in the picture). As the magnetic flux through the loop changes, an alternating current will flow through it, which will have a frequency equal to the frequency of rotation. Then, we need only hook up our wires to get an output alternating current.



Exercise 10. ***Griffiths 9.39.*** For refraction of light from a medium n_2 into a medium with $n_1 < n_2$, Snell's law has a critical angle

$$\theta_c = \arcsin(n_2/n_1).$$

When the incident angle θ_I is greater than θ_c, there is no refracted ray: We get total internal reflection. However, although no energy penetrates the second medium, there is a nonzero field inside the second medium which is rapidly attenuated. We can use the results from class/the textbook with $k_T = \omega n_2/c$ and

$$\mathbf{k}_T = k_T(\sin\theta_T\hat{x} + \cos\theta_T\hat{z}).$$

However, we should now take

$$\sin\theta_T = \frac{n_1}{n_2}\sin\theta_I > 1,$$

so that

$$\cos\theta_T = i\sqrt{\sin^2\theta_T - 1}$$

is imaginary.

a) Show that

$$\tilde{\mathbf{E}}_T(\mathbf{r},t) = \tilde{\mathbf{E}}_{0T}e^{-\kappa z}e^{-i(kx-\omega t)},$$

where

$$\kappa = \frac{\omega}{c}\sqrt{(n_1\sin\theta_I)^2 - {n_2}^2} \quad \text{and} \quad k = \frac{\omega n_1}{c}\sin\theta_I.$$

Notice that this is a wave propagating in the x direction and attenuated in the z direction.

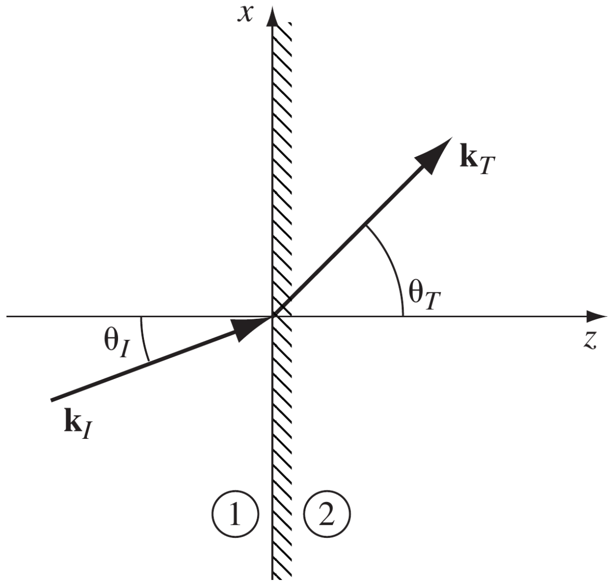
b) Noting that

$$\alpha = \frac{\cos\theta_T}{\cos\theta_I}$$

is now imaginary, use Fresnel's equations

$$\begin{aligned}\tilde{E}_{0R} =& \frac{\alpha - \beta}{\alpha + \beta}\tilde{E}_{0I}\\ \tilde{E}_{0T} =& \frac{2}{\alpha + \beta}\tilde{E}_{0I}\end{aligned}$$

to calculate the reflection coefficient for polarization parallel to the plane of incidence.



c) Do the same for polarization perpendicular to the plane of incidence.

d) In the case of polarization perpendicular to the plane of incidence, show that the real evanescent fields are

$$\begin{aligned}\mathbf{E}(\mathbf{r},t) =& E_0 e^{-\kappa z}\cos(kx-\omega t)\hat{y}\\ \mathbf{B}(\mathbf{r},t) =& \frac{E_0}{\omega}e^{-\kappa z}\left[\kappa\sin(kx-\omega t)\hat{x}+k\cos(kx-\omega t)\hat{z}\right].\end{aligned}$$

e) Check that the fields in (d) satisfy Maxwell's equations.

f) For the fields in (d), construct the Poynting vector, and show that, on average, no energy is transmitted in the z direction.

a) We write

$$\tilde{\mathbf{E}}_T(\mathbf{r},t) = \tilde{E}_{0T}e^{i(\mathbf{k}_T\cdot\mathbf{r}-\omega t},$$

so

$$\tilde{\mathbf{E}}_T(\mathbf{r},t) =\tilde{E}_{0T}\exp[ik_T(x\sin(\theta_T)+z\cos(\theta_T)-\omega t)].$$

We have

$$k_T\sin(\theta_T) = \frac{\omega n_1}{c}\sin(\theta_I)$$

and

$$ik_T\cos(\theta_T) = -\frac{\omega}{c}\sqrt{(n_1\sin\theta_I)^2-n_2{}^2}.$$

Plugging these in to the above formula for $\tilde{\mathbf{E}}$ gives the desired result.

b) We have that

$$R = \left|\frac{\alpha - \beta}{\alpha + \beta}\right|^2 .$$

Since $\alpha = ia$ is imaginary and β is real, we have

$$\frac{ia - \beta}{ia + \beta} = \frac{(ia - \beta)^2}{a^2 + \beta^2},$$

so

$$\begin{aligned} R =& \frac{|\beta^2 - a^2 - 2ia\beta|^2}{(a^2 + \beta^2)^2} \\ =& \frac{(\beta^2 - a^2)^2 + 4a^2\beta^2}{(a^2 + \beta^2)^2} \\ =& 1. \end{aligned}$$

c) Use the result of Griffiths 9.17 from Homework 12. Here, the reflection Fresnel equation reads

$$E_R = \frac{1 - \alpha\beta}{1 + \alpha\beta} E_I .$$

So we have

$$\begin{aligned} R =& \left|\frac{1 - ia\beta}{1 + ia\beta}\right|^2 \\ =& \left|\frac{(1 - ia\beta)^2}{1 + a^2\beta^2}\right|^2 \\ =& \frac{|1 - a^2\beta^2 - 2ia\beta|^2}{1 + a^2\beta^2} \\ =& 1. \end{aligned}$$

d) Take the real part of the result of (a), where we set $\widetilde{\mathbf{E}}_0 = E_0\hat{y}$, to get the result for $\mathbf{E}$. To compute $\mathbf{B}$, we should take the real part of

$$\widetilde{\mathbf{B}} = \frac{1}{c}\hat{k} \times \widetilde{\mathbf{E}}.$$

Note that we can't use the real electric field, since $\mathbf{k}_T$ is itself complex. Taking the cross product, we find

$$\widetilde{\mathbf{B}} = \frac{E_0 k_T}{c} e^{-\kappa z} e^{i(kx - \omega t)} [\sin(\theta_T)\hat{z} - \cos(\theta_T)\hat{x}].$$

Taking the real part, we get

$$\frac{E_0 e^{-\kappa z}}{c} [k_T \sin\theta_T \cos(kx - \omega t)\hat{z} + ik_T \cos\theta_T \sin(kx - \omega t)\hat{x}].$$

Use again that

$$k_T \sin(\theta_T) = \frac{\omega n_1}{c} \sin(\theta_I)$$

and

$$i k_T \cos(\theta_T) = -\frac{\omega}{c} \sqrt{(n_1 \sin\theta_I)^2 - n_2{}^2}.$$

Plugging these in to the above formula gives the desired result for **B**.

e) Notice that the divergence of **E** vanishes easily. For **B**, we have

$$\begin{aligned} \nabla \cdot \mathbf{B} =& \frac{E_0}{\omega} e^{-\kappa z} [\kappa k \cos(kx - \omega t) - \kappa k \cos(kx - \omega t) \\ =& 0. \end{aligned}$$

The curl of **E** gives

$$\nabla \times \mathbf{E} = E_0 e^{-\kappa z} [\kappa \cos(kx - \omega t) \hat{x} - k \sin(kx - \omega t) \hat{z}],$$

which is exactly equal to negative the time derivative of **B**. The final Maxwell's equation reads

$$\nabla \times \mathbf{B} = \frac{n_2{}^2}{c^2} \frac{\partial \mathbf{E}}{\partial t}.$$

The curl of **B** is

$$-\frac{E_0}{\omega} e^{-\kappa z} \sin(kx - \omega t)(\kappa^2 - k^2) \hat{y}.$$

We also have

$$\frac{n_2{}^2}{c^2} \frac{\partial \mathbf{E}}{\partial t} = \frac{n_2{}^2}{c^2} E_0 \omega e^{-\kappa z} \sin(kx - \omega t) \hat{y},$$

so we need to show that

$$k^2 - \kappa^2 = \frac{n_2{}^2 \omega^2}{c^2}.$$

Indeed,

$$k^2 - \kappa^2 = \frac{\omega^2}{c^2} \left(n_1{}^2 \sin^2\theta_I - n_1{}^2 \sin^2\theta_I + n_2{}^2 \right) = \frac{n_2{}^2 \omega^2}{c^2},$$

as desired.

f) We compute

$$\mathbf{S} = \frac{E_0{}^2}{\mu_0 \omega} e^{-2\kappa z} [k \cos^2(kx - \omega t) \hat{x} - \kappa \cos(kx - \omega t) \sin(kx - \omega t) \hat{z}].$$

So we want to compute the time average of the component S_z. This will be proportional to

$$\frac{1}{T} \int_0^T \mathrm{d}t \cos(kx - \omega t) \sin(kx - \omega t) = \frac{1}{2T} \int_0^T \mathrm{d}t \sin[2(kx - \omega t)] = \frac{1}{4\omega T} \cos[2(kx - \omega t)] \Big|_0^T = 0,$$

where by assumption T is a period, so that $\cos(kx - n\omega T) = \cos(kx)$ and $\sin(kx) = \sin(kx - \omega T)$.

Exercise 11. The Dirac Monopole. Consider a half-infinite string of magnetic dipoles, equivalently, a half-infinite solenoid, denoted L.

a) Show that the vector potential outside the string is

$$\mathbf{A}(\mathbf{r}) = -\frac{g}{4\pi}\int_L \mathrm{d}\boldsymbol{\ell} \times \nabla\left(\frac{1}{\mathscr{r}}\right),$$

where g is a constant.

b) Show that the curl of $\mathbf{A}$ is directed radially outward from the end of the string, varies inversely with distance squared from the end of the string, and has total outward flux g.

Remark. The result of (b) shows that the magnetic field outside of the solenoid is that given by a magnetic monopole of exactly charge g. On the other hand, it can be shown (try for yourself!) that changing the position of the string changes $\mathbf{A}$ by a gauge transformation. Explicitly, if we have two different strings L, L', then the integral taken along the closed path $C = L - L'$ will give

$$\mathbf{A}_{L'}(\mathbf{r}) = \mathbf{A}_L(\mathbf{r}) + \frac{g}{4\pi}\nabla\Omega_C(\mathbf{r}),$$

where Ω_C is the solid angle subtended by the contour C at the observation point $\mathbf{r}$. This means that the string itself is not observable, which is consistent with the fact that physical effects due to the monopole should not depend on the theoretical artifice used to create it (the string). In 1930, Dirac famously showed that the existence of magnetic monopoles *implies* the quantization of electric (and magnetic) charge! This is why people have been interested in magnetic monopoles to this day.

a) We can use the dipole term in the multipole expansion for $\mathbf{A}$ to find

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{\mathbf{m}\times\hat{r}}{r^2},$$

for a dipole $\mathbf{m}$ at the origin. In general, for a single dipole along the string,

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{m\hat{z}\times\hat{\mathscr{r}}}{\mathscr{r}^2}.$$

So $\mathbf{A}$ will be the sum or, more precisely, the integral of all such dipoles. Noting that

$$\nabla\left(\frac{1}{\mathscr{r}}\right) = -\frac{\hat{\mathscr{r}}}{\mathscr{r}^2},$$

we can write this integral as

$$\mathbf{A}(\mathbf{r}) = -\frac{\mu_0 m}{4\pi}\int_{-\infty}^{0}\mathrm{d}z\cdot\hat{z}\times\nabla\left(\frac{1}{\mathscr{r}}\right),$$

so that the constant $g = \mu_0 m$, where m is the dipole moment of a single dipole on the string.

b) Take the curl of the integrand, and use “BAC-CAB:”

$$\boldsymbol{\nabla} \times \left(\hat{z} \times \boldsymbol{\nabla}\left(\frac{1}{\mathscr{r}}\right)\right) = \hat{z}\nabla^2\left(\frac{1}{\mathscr{r}}\right) - \frac{\partial}{\partial z}\boldsymbol{\nabla}\left(\frac{1}{\mathscr{r}}\right).$$

Now, the first term is proportional to the delta function

$$\delta^3(\boldsymbol{\mathscr{r}}),$$

which will vanish inside the integral, since $\boldsymbol{\mathscr{r}}$ will never be $\mathbf{0}$ over the integration region (for nonzero $\mathbf{r}$, which we can assume). On the other hand, the second term makes the integral easy:

$$\int_{-\infty}^{0} \mathrm{d}z \frac{\partial}{\partial z}\left(\frac{\hat{\mathscr{r}}}{\mathscr{r}^{2}}\right) = \left.\frac{\hat{\mathscr{r}}}{\mathscr{r}^{2}}\right|_{z=-\infty}^{z=0} = \frac{\hat{r}}{r^2},$$

so the magnetic field is indeed radially outward and varies as $\frac{1}{r^2}$. Let’s compute the flux.

$$\int \frac{\hat{r}}{R^2} \cdot \mathrm{d}\mathbf{a} = 4\pi,$$

where we’ve integrated over a sphere of radius R centered at the origin. Hence, the flux is exactly g.