Week 12 Worksheet
Electrodynamics

Jacob Erlikhman

Exercise 1. An infinite solenoid with a number of wire loops per unit length n is hooked up to an alternating current $I = I_0 \sin(\omega t)$. Find the electric field inside the solenoid if the radius of the solenoid $a \ll c/\omega$. *Hint*: The z-component of the curl in cylindrical coordinates is

$$(\nabla \times \mathbf{v})_z = \frac{1}{s}\left[\frac{\partial}{\partial s}(s v_\varphi) - \frac{\partial v_s}{\partial \varphi}\right].$$

We can find the magnetic field inside the solenoid using Ampère's law as usual. This gives

$$\mathbf{B}_{\text{in}} = \mu_0 n I \hat{z},$$

where z is along the axis of the solenoid. Since

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t},$$

we have that

$$\nabla \times \mathbf{E} = -\mu_0 n I_0 \omega \cos(\omega t).$$

It follows that we can set

$$\mathbf{E} = -\frac{\mu_0}{2} n s I_0 \omega \cos(\omega t)\hat{\varphi}.$$

Note that we can't use a component of $\mathbf{E}$ along $\hat{s}$, since if we do that our solution won't satisfy $\nabla \cdot \mathbf{E} = \mathbf{0}$ (which it must, since there is no charge inside the solenoid).

Alternative Solution: Instead of using Maxwell's equation

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

in its infinitesimal form, we could integrate both sides over a region:

$$\int \nabla \times \mathbf{E} \cdot \mathrm{d}\mathbf{a} = -\frac{\partial \Phi_B}{\partial t},$$

where Φ_B is the magnetic flux through the region of integration. Now, apply Stokes' theorem to get

$$\oint \mathbf{E} \cdot \mathrm{d}\boldsymbol{\ell} = -\frac{\partial \Phi_B}{\partial t}.$$

Using a loop that is concentric with the cylinder (and inside it) gives

$$2\pi s E_\varphi = -\mu_0 n I_0 \omega \cos(\omega t) \pi s^2,$$

since E_φ must be independent of φ by symmetry. Hence,

$$E_\varphi = -\frac{\mu_0}{2} n s I_0 \omega \cos(\omega t),$$

which is the same answer we got above. We still need to show that $E_s = 0$. I think for this we need the form of the divergence for cylindrical coordinates, namely that

$$\nabla \cdot \mathbf{E} = \frac{1}{s}\frac{\partial}{\partial s}(s E_s),$$

where the other two terms vanish. From here, since $\nabla \cdot \mathbf{E} = 0$, we have that this must be 0, which can only hold if $E_s \sim \frac{1}{s}$. But this isn't a valid solution, since it blows up at the origin. Hence, $E_s = 0$.

Exercise 2. A toroidal coil has N tightly wound turns of wire with current $I(t) = kt$, where k is a constant. The torus has outer radius a (in the equatorial plane), and it has a circular cross section of radius $b \ll a$. Find the following fields, ignoring any components of the current which are perpendicular to the circular cross sections.

a) The magnetic field inside and outside the solenoid.

b) The electric field at a distance $r \gg a$ from the coil. Your answer can be of the form of the lowest multipole moment (e.g. monopole moment, dipole moment, etc.).
Hints: Maxwell's equations for Faraday fields are identical to those for *magnetostatics*, with $\mathbf{E}$ and $-\partial_t \mathbf{B}$ playing the role of $\mathbf{B}$ and $\mu_0 \mathbf{J}$, respectively.

a) This is the same as for a constant current: $\mathbf{B} = \mu_0 n I \hat{\varphi}$ inside and $\mathbf{0}$ outside, where $n = N/2\pi a$.

b) Using the hint, we deduce that far away $\mathbf{E}$ will behave as a *magnetic* dipole field, since there are no magnetic monopoles. Thus, we can compute its magnetic dipole moment to be

$$m = I_B A,$$

where A is the area of the toroidal loop of wire. But be careful: The I_B in this formula is *not* the current; rather, it is the time derivative of the magnetic flux,

$$I_B = \frac{1}{\mu_0}\partial_t \Phi_B.$$

Thus, we can compute

$$I_B = nk \cdot \pi b^2,$$

and

$$A = \pi a^2.$$

Thus, we find

$$m = \frac{Nk\pi b^2 a}{2}.$$

We could then plug this in to the formula from the book for the field of a magnetic dipole to deduce the electric field. Note that the electric field due to such a "magnetic" dipole will actually be the same as that due to an *electric* dipole of the same magnitude!