

Week 14 Worksheet

Waves and Energy

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Exercise 1. a) Starting with Maxwell's equations in vacuum, show that they give you wave equations for \mathbf{E} and \mathbf{B} .

b) Show that the waves are transverse: If they travel in the z direction, then $\tilde{E}_{0z} = 0$ and $\tilde{B}_{0z} = 0$.

c) Show that \mathbf{B} is perpendicular to \mathbf{E} .

Exercise 2. Recalling that magnetic forces do no work, we have that

$$dW = \mathbf{F} \cdot d\boldsymbol{\ell} = q\mathbf{E} \cdot \mathbf{v} dt.$$

Thus, the rate at which work is done on the charges in a volume is

$$\frac{dW}{dt} = \int \mathbf{E} \cdot \mathbf{J} dV.$$

a) Show that

$$\mathbf{E} \cdot \mathbf{J} = -\frac{1}{2} \frac{\partial}{\partial t} \left(\epsilon_0 E^2 + \frac{1}{\mu_0} B^2 \right) - \frac{1}{\mu_0} \nabla \cdot \mathbf{E} \times \mathbf{B}$$

Hints: Use Maxwell's equations along with the identity

$$\nabla \cdot (\mathbf{v} \times \mathbf{w}) = \mathbf{w} \cdot \nabla \times \mathbf{v} - \mathbf{v} \cdot \nabla \times \mathbf{w}.$$

b) Plug this in to the formula for the rate of work, and derive Poynting's theorem:

$$\mathbf{S} = \frac{1}{\mu_0} \mathbf{E} \times \mathbf{B}$$

is the energy per unit time, per unit area, transported by the fields; it is the energy flux density.

c) You showed in class that

$$\begin{aligned} \mathbf{g} &= \mu_0 \epsilon_0 \mathbf{S} \\ &= \frac{1}{c^2} \mathbf{S} \end{aligned}$$

is the momentum per unit volume stored in the fields. Discuss why \mathbf{S} seems to have two different physical interpretations.

Both of these exercises are done in Griffiths. The only comment I will make is for part (c) of the second exercise. Recall from special relativity

$$E^2 = p^2 c^2 + m^2 c^4.$$

Since $m = 0$ for an electromagnetic wave, we have the relation

$$E = pc.$$

So there is a proportionality relation between energy and momentum for light. Hence, when we say we have an energy density, we also have a momentum density directly from this formula. It is now straightforward to check that the units work out correctly: Since S has units of energy per unit area per unit time,

$$[S] = \frac{E}{L^2 T},$$

this is the same by $E = pc$ as

$$[S] = \frac{M}{T^3},$$

where M denotes units of mass. Now, a momentum per unit volume has units of

$$\frac{M}{L^2 T},$$

and we notice that this is exactly a factor of c^2 off from $[S]$. Another comment is that this is physically sensible. Since the energy is carried out by the fields, and the fields travel at speed c , it follows that we get an extra factor of c when we convert from \mathbf{S} to \mathbf{g} . So one factor comes from $E = pc$, and the other comes from the fact that the wave itself travels at speed c .