Week 15 Worksheet Solutions
EM Waves in Matter

Jacob Erlikhman

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Exercise 1. A plasma in a uniform magnetic field $\mathbf{B} = B\hat{z}$ is a **gyrotropic material**, which means it has an anisotropic dielectric tensor $\varepsilon = (\varepsilon_{ij})$ with nonzero components $\varepsilon_{11} = \varepsilon_{22} = \varepsilon_1$, $\varepsilon_{12} = -\varepsilon_{21} = i\varepsilon_2$, $\varepsilon_{33} = \varepsilon_3$, where the ε_i are real numbers (in the absence of losses). Assume that the plasma has $\mu = \mu_0$ and that there are no free charges or currents.

a) Show that the equation $\mathbf{D} = \varepsilon\mathbf{E}$ for such a material can be written in the form

$$\mathbf{D} = \varepsilon_\parallel E_z\hat{z} + \varepsilon_\perp\mathbf{E}_\perp + i\mathbf{E}\times\mathbf{g},$$

where $\mathbf{g} = g\hat{z}$ is the **gyration vector** and $\mathbf{E}_\perp$ is the part of $\mathbf{E}$ perpendicular to $\mathbf{g}$. Make sure you determine $\varepsilon_\parallel$, $\varepsilon_\perp$, and g in terms of the components of ε.

b) Look for solutions to Maxwell's equations with wave vector $\mathbf{k} = k\hat{z}$. Show that Maxwell's equations reduce to a single eigenvalue-eigenvector equation for $\mathbf{E}_\perp$.
Hint: Show first that $E_z = 0$.

c) Solve the equation from (b), and determine the allowed magnitudes of the wavevector $k_\pm$.

d) Determine the type of polarization (in the sense of Exercise 2, below) of your solutions to (b) and (c).

a) This is just matrix multiplication. You should find that

$$\begin{aligned}\mathbf{D} &= \begin{bmatrix}\varepsilon_1 & i\varepsilon_2 & 0\\ -i\varepsilon_2 & \varepsilon_1 & 0\\ 0 & 0 & \varepsilon_3\end{bmatrix}\mathbf{E}\\ &= \begin{bmatrix}\varepsilon_1 E_x + i\varepsilon_2 E_y\\ -i\varepsilon_2 E_x + \varepsilon_1 E_y\\ \varepsilon_3 E_z\end{bmatrix}\\ &= \varepsilon_3 E_z\hat{z} + i\varepsilon_2\mathbf{E}\times\hat{z} + \varepsilon_1\mathbf{E}_\perp.\end{aligned}$$

b) First, use the Maxwell equation

$$\nabla\cdot\mathbf{D} = 0.$$

Plugging in the form for **D** from (a), we find

$$\varepsilon_3 \partial_z E_z = 0,$$

since the x- and y-derivatives of **E** vanish. This implies that $E_z = 0$, since we assume that **E** is a plane wave (hence can't be constant). In particular, this means that $\nabla \cdot \mathbf{E} = 0$ for plane wave solutions, too. Now, take the curl of the equation

$$\nabla \times \mathbf{E} = i\omega \mathbf{B},$$

to find

$$\nabla^2 \mathbf{E} = \omega^2 \mu_0 \mathbf{D},$$

where on the RHS we use the equation

$$\nabla \times \mathbf{B} = -i\omega\mu_0 \mathbf{D}.$$

Plugging in the form for **D** from part (a) into the RHS of the equation for **E**, we find

$$-k^2 \mathbf{E} = \omega^2 \mu_0 \left[i\varepsilon_2 (E_y \hat{x} - E_x \hat{y}) + \varepsilon_1 \mathbf{E} \right],$$

where

$$\mathbf{E} = \mathbf{E}_\perp = \begin{bmatrix} E_x \\ E_y \\ 0 \end{bmatrix}.$$

We can write this a system of linear equations:

$$\frac{k^2}{\mu_0 \omega^2} \mathbf{E} = \varepsilon \mathbf{E},$$

and this is the eigenvalue-eigenvector system we wanted.

c) Noting that the z-component of $\mathbf{E} = \mathbf{E}_\perp$ is zero, we are looking for the eigenvalues and eigenvectors of

$$\begin{bmatrix} \varepsilon_1 & i\varepsilon_2 \\ -i\varepsilon_2 & \varepsilon_1 \end{bmatrix}.$$

We find the eigenvalues are

$$\varepsilon_1 \pm \varepsilon_2$$

with corresponding eigenvectors

$$\begin{bmatrix} 1 \\ -i \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 \\ i \end{bmatrix}.$$

Thus, we find that

$$\frac{k_\pm^2}{\mu_0\omega^2} = \varepsilon_1 \pm \varepsilon_2.$$

Solving this for k, we have

$$k_\pm = \omega\sqrt{\mu_0(\varepsilon_1 \pm \varepsilon_2)}.$$

d) Since the eigenvectors are $(1, i)$ and $(1, -i)$, we see that the relative phases of the x- and y-components of $\mathbf{E}$ are precisely $\pi/2$. This is circular polarization. So the allowed modes along the z-direction are two waves which are left- and right-circularly polarized, respectively, with fixed wave vector magnitudes $k_\pm$ above. Note that $|\omega/k|$ defines the speed of the wave in this medium, so that there are two allowed speeds,

$$\frac{1}{\sqrt{\mu_0(\varepsilon_1 \pm \varepsilon_2)}}.$$

Exercise 2. Polarization. When we solve Maxwell's equations in vacuum, we end up with monochromatic plane wave solutions (traveling in the z-direction) of the form

$$\mathbf{E} = (E_1\hat{x} + E_2\hat{y})e^{i(kz-\omega t)},$$

where the E_i are in general complex numbers. If the phases of E_i are *the same*, we say that the wave is **linearly polarized**. If the phases of E_i differ by $\pi/2$, we say the wave is **circularly polarized**.

a) Explain why the name "linear polarization" makes sense.

b) For linearly polarized $\mathbf{E}$, what is $\mathbf{B}$? Is it also linearly polarized?

c) Do the same for "circular polarization."

d) What would you call "elliptical polarization?" Why?

a) If the wave is linearly polarized, then the real part of $\mathbf{E}$ (which is the physical electric field) will be in the (x, y)-plane at an angle $\alpha = \arctan(E_2/E_1)$ to the x-axis and have a magnitude $\sqrt{|E_1|^2 + |E_2|^2}$. This is the same direction and magnitude for all times and everywhere, so it is linear (in one direction).

b) We know that $\mathbf{B} \propto \hat{z} \times \mathbf{E}$ from last week's worksheet. So $\mathbf{B}$ will be orthogonal to $\mathbf{E}$ and also linearly polarized.

c) In the case of circular polarization, the magnitudes of the components of the real electric field are

$$\begin{aligned} E_x &= E_0 \cos(kz - \omega t) \\ E_y &= \mp E_0 \sin(kz - \omega t). \end{aligned}$$

Thus, we see that the electric field rotates in the (x, y)-plane with frequency ω. The magnitude however remains constant (equal to E_0^2). So the vector of the physical electric field traces out a *circle* in the (x, y)-plane. Same for $\mathbf{B}$.

d) If the relative phase of E_i is $\delta \in (0, \pi/2)$ and $|E_1| = |E_2| = E_0$, then we end up with elliptical polarization. In this case, the physical electric/magnetic fields will trace out an ellipse in the (x, y)-plane. We see that the components of the physical electric field are given by

$$E_x = E_0[\cos(kz - \omega t)\cos(\delta/2) - \sin(kz - \omega t)\sin(\delta/2)]$$
$$E_y = E_0[\cos(kz - \omega t)\cos(\delta/2) + \sin(kz - \omega t)\sin(\delta/2)].$$

This traces out an ellipse in the (x, y)-plane. Indeed, an ellipse is an affine transformation of a circle (see ✎ **here**). In our case, the matrix that we're transforming the circle with radius E_0 by is

$$\begin{bmatrix} \cos(\delta/2) & -\sin(\delta/2) \\ \cos(\delta/2) & \sin(\delta/2) \end{bmatrix}.$$

This gives the semi-axes

$$E_0\left(\sqrt{\frac{1+\sin\delta}{2}} \pm \sqrt{\frac{1-\sin\delta}{2}}\right)$$