Week 2 Worksheet
Math Review

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Exercise 1.

a) What does the gradient tell you about a function? Why?
Hints: If $\nabla f(\mathbf{x}) = \mathbf{w}$, argue or show that

$$D_{\hat{v}} f(\mathbf{x}) = \hat{v} \cdot \mathbf{w},$$

where $D_{\hat{v}} f(\mathbf{x})$ is the directional derivative of f at $\mathbf{x}$ in the direction $\hat{v}$. It may help to consider the special cases $\hat{v} = (0, 1)$ and $\hat{v} = (1, 0)$ in the case that $f(x, y)$ is a function on the plane. In this scenario, what do you get if $\hat{v} = (a, b)$ (with $a^2 + b^2 = 1$)?

Remark. Notice that this result holds in *any dimension* $n \in \mathbb{N}$.

b) What does the curl tell you about a vector field? Why?
Hint: Draw and calculate the curls of some example vector fields, like $-y\hat{x} + x\hat{y}$ or $x\hat{y}$. Now, try the vector fields $x\hat{x} + y\hat{y} + z\hat{z}$, $\hat{z}$, and $z\hat{z}$.

c) What does the divergence tell you about a vector field? Why?
Hint: Using the same vector fields from the hint from (b), calculate the divergence for each of them.

d) Use (a) and (b) to give an intuitive explanation of why the curl of a gradient is always 0.

e) Use (b) and (c) to give an intuitive explanation of why the divergence of a curl is always 0.

f) Show that $\nabla \times \nabla f = 0$ directly.

g) Show that $\nabla \cdot \nabla \times \mathbf{v} = 0$ directly.

Exercise 2. ***Griffiths 1.13.*** Let $\mathbf{d}$ be the separation vector from a fixed point (x', y', z') to the point (x, y, z), and let d be its length. Show that

a) $\nabla(d^2) = 2\mathbf{d}$,

b) $\nabla(1/d) = -\hat{d}/d^2$.

c) What is the general formula for $\nabla(d^n)$?

d) You computed these formulas in cartesian coordinates. Do they hold in other coordinate systems? Why or why not?

Remark. To prove this would require quite a bit of work or more tools than we have at our disposal. However, you should be able to come up with an intuitive argument.

Exercise 3. Challenge/Extra Problem. The Stokes' and divergence theorems have generalizations to any dimension $n \in \mathbb{N}$. In this problem, you'll get an idea of what those are.

a) The divergence theorem directly generalizes to any dimension. Write down the generalization.

b) Focus for the divergence theorem on the 1- and 2-dimensional cases (i.e. the "volume" we're integrating over is 1- or 2-dimensional). Do either of these look familiar?

c) To generalize Stokes' theorem requires more tools than we have at our disposal at the moment. However, we can readily "specialize" to dimension 1. Write down what Stokes' theorem should say in dimension 1.
Hint: Use the usual Stokes' theorem. Consider an integration region of the form $(-\varepsilon, \varepsilon) \times [a, b]$, and take $\varepsilon \to 0$.

d) Now, specialize to dimension 2. What do you obtain? It should look similar to your 2-dimensional result from (b).

e) Explain why in dimensions 1 and 2 the divergence and Stokes' theorems give the same results.

Remark. Although it isn't clear from this exercise, it is actually the generalized Stokes' theorem that is (more of) a generalization of the fundamental theorem of calculus, not the divergence theorem.