Week 5 Worksheet
Multipole Expansion and Boundary Value Problems

Jacob Erlikhman

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Exercise 1.

a) ***Griffiths 3.52a.*** Show that the quadrupole term in the multipole expansion,

$$V_{\text{quad}}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0 r^3} \int r'^2 \left(\frac{3}{2}\cos^2\alpha - \frac{1}{2} \right) \rho(\mathbf{r}')\, \mathrm{d}V',$$

where α is the angle between $\mathbf{r}$ and $\mathbf{r}'$, can be written

$$V_{\text{quad}}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0 r^3} \sum_{i,j=1}^{3} \hat{r}_i \hat{r}_j Q_{ij},$$

where $\hat{r}_i \hat{r}_j Q_{ij} = \hat{r} \cdot (Q\hat{r})$ and

$$Q_{ij} = \frac{1}{2} \int [3r'_i r'_j - (r')^2 \delta_{ij}] \rho(\mathbf{r}')\, \mathrm{d}V'.$$

We call Q_{ij} the **quadrupole moment** of the charge distribution. Notice that the monopole moment Q is a scalar, the dipole moment $\mathbf{p}$ is a vector, the quadrupole moment Q_{ij} is a rank 2 tensor (i.e. a matrix), and so on.

b) At the center of a line of charge of length 2ℓ with linear charge density λ is placed a point charge $q = -2\lambda\ell$. Find the quadrupole moment of this system and the potential at large distances.

Exercise 2. Two infinite, solid conducting cones have common axis (z), common vertex (O), and equal opening angles 2α (i.e. the equations which define the surfaces of the cones in spherical coordinates are $\theta = \alpha$ and $\theta = \pi - \alpha$, respectively). The potential difference between the cones is V_0, and they are electrically insulated from each other.

a) Start with the azimuthally symmetric laplacian in spherical coordinates:

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right)+\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right),$$

and find the θ equation by separation of variables.
Hint: Write the r equation as

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}R}{\mathrm{d}r}\right)=\ell(\ell+1)R.$$

b) Recall that the equation has a set of solutions given by the Legendre polynomials, but this is *only one solution to the equation*. Since we have a second order equation, there should be another one. It is given by $\ln(\tan(\theta/2))$. Check that it satisfies the equation from (a). For what values of θ is this solution valid?

c) Find the potential and electric field in the region between the cones.
Hints: As usual, we want to match the boundary conditions to the general form of the solution. Writing the Legendre polynomials as $P_\ell(\cos\theta)$, the usual general form that we use can be written $\sum_\ell R_\ell(r)P_\ell(\cos\theta)$, where R_ℓ are the solutions to the radial equation from part (a).