# Week 6 Worksheet Solutions
# Polarization and Dielectrics

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**Exercise 1.**

a) Consider a line of dipoles of equal moment $qd$, where the end of one touches the beginning of the next. Suppose the line is of length $\ell$. What is the dipole moment of the entire line?

b) Now, suppose you had a cylinder of length $\ell$, cross-sectional area $A$, and uniform polarizaiton $\mathbf{P}$, which is parallel to the axis of the cylinder. What is the surface charge density on the ends of the cylinder?

c) Argue that the bound surface charge for any uniformly polarized object is

$$\sigma_b = \mathbf{P} \cdot \hat{n}.$$

*Hint*: Consider the cylinder as in (b), with axis along the $x$-axis, but this time with one of the ends cut at an angle $\theta$ relative to the $y$-axis.

d) What happens if the polarization is *nonuniform*? Consider a sphere with a diverging $\mathbf{P}$. The charge which accumulates at the edges must be $\sigma_b$, so what is the net charge inside the sphere?

e) Argue that

$$\int \rho_b \, dV = -\oint \mathbf{P} \cdot d\mathbf{a},$$

and use the divergence theorem to obtain that $\rho_b = -\nabla \cdot \mathbf{P}$ for the sphere of (d). Argue that this holds for any object.

   Although this is done in Griffiths, I think he doesn't give too many details, so I'll provide my own solutions to this exercise.

a) Where one dipole meets the next, there is a charge of magnitude $q$ and another of magnitude $-q$, so they cancel each other out. Thus, the whole line just because two charges at the ends, so the dipole moment is $p = q\ell$.

b) We can model the cylinder as a bunch of lines of dipoles as in (a). But now, the dipole moment of each is $\sigma_b \ell$, since each one contributes an infinitesimal charge element of the cap of the cylinder. So the total dipole moment is $p = \sigma_b \ell A$, and, since the polarization is uniform,

$$P = \frac{p}{A\ell}$$
$$= \sigma_b.$$

c) We need to compare the area of the cut cap of the cylinder to the uncut cap. Since it's cut at an angle $\theta$ to the $y$-axis, we see that the cut section is an ellipse, where the semi-minor axis has length $r$ and the semi-major axis has length $r/\cos(\theta)$. The area of the ellipse is then

$$A_e = \pi \frac{r^2}{\cos\theta},$$

which makes sense since it should be *larger* than the area of a circle of radius $r$. OK, so what's $\sigma_b$? Well if the old $\sigma_b$ was $P$, then the new one should be *smaller* than the old one (since $\sigma_b A_e = Q$ is the charge on the ellipse). Thus,

$$\sigma_b = P\cos\theta,$$

since we know that $\sigma_b A_e$ should also be the charge on the uncut portion of the cylinder. This confirms the formula $\mathbf{P} \cdot \hat{n} = \sigma_b$ in this case. Why does it hold in general? Since $\sigma_b$ is the charge *density*, we are free to zoom in to an infinitesimal chunk of the boundary of our dielectric material to calculate it. Although this boundary may be curved, it won't be on this infinitesimal chunk, so this model of a cut cylinder will apply there. We then glue the chunks together to get all of $\sigma_b$.

d) If the polarization is nonuniform, some charge will accumulate inside the sphere. If the charge which accumulates on the boundary is $Q$, then the charge which accumulates inside must be $-Q$, since the material was neutral to begin with.

e) We have

$$-Q = -\int \sigma_b da = -\int \mathbf{P} \cdot \hat{n} \, da = -\int \mathbf{P} \cdot d\mathbf{a},$$

where the integral is taken over the boundary of the material. But we also have that

$$-Q = \int \rho_b d^3 r,$$

where the integral is taken over the bulk of the material. Finally, using Gauss' law, we have

$$-\int \mathbf{P} \cdot d\mathbf{a} = -\int \nabla \cdot \mathbf{P} d^3 r,$$

where the integral is again taken over the bulk. Thus, we find

$$\int \rho_b d^3 r = -\int \nabla \cdot \mathbf{P} d^3 r,$$

so that $\rho_b = -\nabla \cdot \mathbf{P}$.

**Exercise 2.** A dielectric of arbitrary shape, volume $V$, and relative permittivity $\varepsilon_r$ which is close to 1 (i.e. such that $\varepsilon_r - 1 \ll 1$) is brought into a uniform electric field $\mathbf{E}$. Outside the dielectric, $\varepsilon_r = 1$. Find the field at a large distance $r$ from the dielectric.

Then we get that the polarization is

$$\mathbf{P} = (\varepsilon_r - 1)\mathbf{E},$$

so that we have an induced dipole moment

$$\mathbf{p} = \int \mathrm{d}V\,\mathbf{P}$$
$$= V(\varepsilon_r - 1)\mathbf{E}.$$

The field at large distances will then be the field due to such a dipole moment plus the external field. The former is given by (Griffiths Equation 3.103)

$$\mathbf{E}_{\mathrm{dip}} = \frac{p}{4\pi\varepsilon_0 r^3}(2\cos(\theta)\hat{r} + \sin(\theta)\hat{\theta}),$$

where we take $\mathbf{E}$ to point in the $z$ direction (so that $\mathbf{p}$ is in that direction as well).