

# Week 9 Worksheet Solutions

## Magnets!

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**Exercise 1.** The Lorentz force law for a particle of charge  $q$  moving with velocity  $\mathbf{v}$  in a magnetic field  $\mathbf{B}$  is given by

$$\mathbf{F}_{\text{mag}} = q\mathbf{v} \times \mathbf{B}.$$

Find an expression for the work that the magnetic force does on the particle.

The work is 0, since  $d\ell = \mathbf{v}dt$ , so

$$dW = \mathbf{F} \cdot d\ell = q\mathbf{v} \times \mathbf{B} \cdot \mathbf{v}dt = 0.$$

**Exercise 2.** Find the magnetic field and vector potential due to a current which flows with constant surface density  $\mathbf{K}$  which flows along the surface of an infinite cylinder of radius  $a$  in the following directions:

- along the axis of the cylinder;
  - perpendicular to the axis of the cylinder;
  - at an angle  $\alpha$  to the axis of the cylinder.
- a) The magnetic field can be found using Ampère's law. Clearly, the field will be along  $\hat{\phi}$  if the axis of the cylinder is along the  $z$ -axis. Ampère's law then gives that the field will be 0 inside the cylinder. Outside, it will be given by

$$\mu_0 K \cdot 2\pi a \ell = \ell B(s) 2\pi s \implies \mathbf{B}(s) = \frac{\mu_0 K a}{s} \hat{\phi}.$$

To find the vector potential, use

$$\oint \mathbf{A} \cdot d\ell = \int \mathbf{B} \cdot d\mathbf{a},$$

which you can obtain from Stokes' theorem and the fact that  $\mathbf{B} = \nabla \times \mathbf{A}$ . Also, since  $\mathbf{B}$  is the curl of  $\mathbf{A}$ , we see that if  $\mathbf{B}$  is along  $\hat{\phi}$ , then  $\mathbf{A}$  will be along  $\hat{z}$ . It follows that outside the cylinder we can use

a rectangular amperian loop which is placed so that  $\mathbf{B}$  is perpendicular to it while  $\mathbf{A}$  is parallel to two of its sides (and perpendicular to the other two). This loop will then give that

$$A(s_1) - A(s_2) = \mu_0 K a \ln\left(\frac{s_2}{s_1}\right).$$

If we use a loop such that  $s_1$  is inside the cylinder while  $s_2$  is outside, we can, noting that  $A$  must be constant inside the cylinder, set

$$\mathbf{A}(s) = -\mu_0 K a \ln(s) \hat{z}$$

outside the cylinder. Notice that this agrees with everything:  $\nabla \times \mathbf{A} = \mathbf{B}$ ,  $\nabla \cdot \mathbf{A} = 0$ , and it satisfies the relation above. Inside the cylinder, we can set

$$\mathbf{A}(s) = -\mu_0 K a \ln(a) \hat{z},$$

which clearly satisfies the requirements (and gives us a continuous function for  $\mathbf{A}$ ).

- b) This situation is the same as a solenoid. As in that case, the magnetic field is 0 outside (since it has to go to 0 as  $s \rightarrow \infty$  and an amperian loop gives that  $B(s_1) = B(s_2)$  for  $s_i > a$ ). Inside, we instead have that

$$B\ell = \mu_0 K \ell$$

by Ampère's law, so

$$\mathbf{B}(s) = \mu_0 K \hat{z}$$

inside (note that this is constant). Since  $\mathbf{B}$  is the curl of  $\mathbf{A}$ , we have as in (a) that if  $\mathbf{B}$  is along  $\hat{z}$ ,  $\mathbf{A}$  will be along  $\hat{\phi}$ . Using this, we can proceed as in (a) with an amperian loop inside the solenoid that circles around the  $z$ -axis. This will give

$$\mu_0 K \cdot \pi s^2 = A \cdot 2\pi s,$$

so that

$$\mathbf{A}(s) = \frac{\mu_0 K s}{2} \hat{\phi}$$

inside the solenoid. On the other hand, using an amperian loop with radius  $s > a$ , we have

$$\mu_0 K \cdot \pi a^2 = A \cdot 2\pi s,$$

so

$$\mathbf{A}(s) = \frac{\mu_0 K a^2}{2s} \hat{\phi}.$$

Note that this form for  $\mathbf{A}$  is continuous at the boundary  $s = a$ , i.e.  $\mathbf{A}_{\text{in}}(a) = \mathbf{A}_{\text{out}}(a)$ . You can also check that  $\nabla \times \mathbf{A} = \mathbf{B}$  everywhere and  $\nabla \cdot \mathbf{A} = 0$  everywhere.

c) In this case our  $\mathbf{K}$  can be written as

$$\mathbf{K} = K_{\perp} \hat{\phi} + K_{\parallel} \hat{z},$$

where

$$\begin{aligned} K_{\perp} &= K \sin \alpha \\ K_{\parallel} &= K \cos \alpha. \end{aligned}$$

Now, since *everything is linear*, the solutions in this case will be linear combinations of the solutions in parts (a) and (b), with the appropriate values of  $K$ . This statement means that 1.  $\mathbf{B} = \nabla \times \mathbf{A}$  is linear, 2.  $\nabla \cdot \mathbf{A} = 0$  is linear, and 3.  $\mathbf{B}$  satisfies the principle of superposition. Hence, our answers are

$$\begin{aligned} \mathbf{B}_{\text{out}}(s) &= \mu_0 K_{\parallel} \frac{a}{s} \hat{\phi} \\ \mathbf{B}_{\text{in}}(s) &= \mu_0 K_{\perp} \hat{z} \\ \mathbf{A}_{\text{in}}(s) &= \frac{\mu_0 K_{\perp} s}{2} \hat{\phi} - \mu_0 K_{\parallel} a \ln(a) \hat{z} \\ \mathbf{A}_{\text{out}}(s) &= \frac{\mu_0 K_{\perp} a^2}{2s} \hat{\phi} - \mu_0 K_{\parallel} a \ln(s) \hat{z}. \end{aligned}$$