

Midterm Review

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October 17, 2026

Exercise 1 (*A free electron cannot absorb a photon*). In this problem, you will show that a free electron cannot absorb a photon and conserve both energy and momentum (hence, the photoelectric effect requires a *bound* electron).

Hint: Nothing is lost by working in the electron's rest frame, since the electron is massive and a process forbidden in one inertial frame is forbidden in all. In that frame the electron starts at rest while the photon has momentum p and energy $E_\gamma = pc$; recall $E_e = \sqrt{p^2c^2 + m^2c^4}$ for an electron of momentum p and mass m .

- Using momentum conservation, find the momentum and the energy of the electron just after it absorbs the photon.
- Now impose energy conservation (initial energy of the system equals final energy). Show that the two requirements can hold simultaneously only if $p = 0$, i.e. only if there was no photon at all, so a free electron cannot absorb a photon.

In the electron rest frame, only the photon has initial momentum p_i . Since the photon is absorbed, the final momentum is only the electron's; call it p_f . momentum conservation says $p_i = p_f \equiv p$. The initial energy is the sum of the photon energy and the electron energy. Since the electron is at rest, its energy is just $E_e = m$; on the other hand, the photon energy is $E_\gamma = p$, since the photon is massless. Thus, $E_i = m + p$. The final energy is only the electron's, which is $E_f = \sqrt{m^2 + p^2}$. Energy conservation equates $E_i = E_f$, so we get

$$m + p = \sqrt{m^2 + p^2}.$$

Squaring both sides of this equation, we find

$$2mp = 0,$$

so $p = 0$. This means there was no photon to begin with!

Exercise 2 (*Bohr Radius*). Let a be an estimate of the size of the hydrogen atom, so that the electron is confined to a distance of order a .

- Write down an expression for the momentum of the electron (order of magnitude estimate is fine).
- Supposing the orbit is circular, determine the centrifugal force on the electron.

- c) Equate this to the electrostatic force on the electron due to the proton, and solve for a to obtain

$$a = \frac{\hbar^2}{me^2},$$

which is the Bohr radius.

- a) We use the uncertainty relation $\sigma_x \sigma_p \sim \hbar$ and $\sigma_x \sim a$ to see that $\sigma_p \sim p \sim \hbar/a$.
- b) The centrifugal force of a mass m with momentum p in a circular orbit of radius a is p^2/ma , so the force is $\hbar^2/ma^3$.
- c) The electrostatic force is given by Coulomb's law,

$$F = \frac{e^2}{a^2}.$$

Setting this equal to $\hbar^2/ma^3$ and solving for a we get

$$a = \frac{\hbar^2}{me^2}$$

as desired.

Exercise 3. Show that if $E < V(x)$ for all x , then $\psi(x) \equiv 0$.

We check

$$\begin{aligned} E &= \langle H \rangle \\ &= \left\langle \frac{p^2}{2m} + V(x) \right\rangle \\ &= \frac{1}{2m} \langle p^2 \rangle + \langle V \rangle \\ &= \frac{1}{2m} \langle p\psi | p\psi \rangle + \langle V \rangle. \end{aligned}$$

The first term is nonnegative, because it is proportional to $\int |p\psi|^2 dx$. On the other hand, the second term cannot be less than the minimum value of $V(x)$, which means neither can E .

Exercise 4. Suppose a free particle moves not on the x -axis but on a circle of radius R .

- a) Find the stationary states and the allowed energies.
- b) You should have found in (a) that there are degenerate states (i.e. linearly independent stationary states with the same energy). How is that consistent with the result of Exercise 2 of the Week 4 discussion section worksheet?

- a) We need to solve the TISE with the periodic boundary conditions $\psi(0) = \psi(2\pi R)$. Set $2\pi R = L$. Then the free particle TISE is just

$$-\frac{\hbar^2}{2m}\psi'' = E\psi,$$

which has solutions

$$\psi(x) = Ae^{ikx} + Be^{-ikx}.$$

Enforcing the periodic boundary conditions, we find

$$A + B = Ae^{ikL} + Be^{-ikL}.$$

Similarly, the equation for the first derivative gives

$$A - B = Ae^{ikL} - Be^{-ikL}.$$

Adding these two equations together, we see that

$$e^{ikL} = 1,$$

so that $k = 2n\pi/L$. Unlike in the case of the free particle on the real line, these solutions *are* normalizable. Indeed, we see that we get two linearly independent solutions given by "right-moving" waves

$$\psi_+(x) = \frac{1}{\sqrt{L}}e^{2in\pi x/L}$$

and "left-moving" waves

$$\psi_-(x) = \frac{1}{\sqrt{L}}e^{-2in\pi x/L}.$$

- b) In order to use the result of the cited exercise, we need to check that $\psi_-(x) = \psi_+(x) = 0$ for some value of x , i.e. we need to check that

$$e^{i\alpha x} = e^{-i\alpha x} = 0$$

for some x . But exponentials are never 0, so this is impossible. Hence, there is no contradiction.

Exercise 5. Let $f(x)$ be an arbitrary function of position x and $g(p)$ be an arbitrary function of momentum p . Evaluate the commutators $[p, f(x)]$ and $[x, g(p)]$.

Hint: To compute the second commutator, you will want to first show that $x = i\hbar d/dp$.

We write $p = -i\hbar d/dx$ and use a test function $\psi(x)$. Thus, we compute

$$[p, f(x)]\psi(x) = p(f\psi) - fp\psi = \psi pf + fp\psi - fp\psi = \psi pf.$$

Dropping, the test function ψ , we see

$$[p, f(x)] = pf(x) = -i\hbar f'.$$

To see that $x = i\hbar d/dp$, we make the following observation. As before, we want to test

$$[x, p]\varphi(p) = i\hbar\varphi(p).$$

Now, notice that if we start with $[p, x]\varphi(x)$, plug in $p = -i\hbar d/dx$, and replace $p \leftrightarrow x$, we get

$$[-i\hbar d/dx, x]\varphi(x) = -i\hbar\varphi(x) \leftrightarrow [-i\hbar d/dp, p]\varphi(p) = -i\hbar\varphi(p).$$

Since we know $[x, p] = i\hbar$, not $-i\hbar$, we deduce that $x = i\hbar d/dp$. Now, we have

$$[x, g(p)]\psi = x(g(p)\psi) - g(p)x\psi = xg(p)\psi,$$

where we use the product rule for the first term $x(g(p)\psi)$. As before, we drop the test function and use $x = i\hbar d/dp$ to see

$$[x, g(p)] = xg(p) = i\hbar g'(p).$$

Exercise 6. A particle of mass m sits in the ground state of an infinite square well with walls at $x = 0$ and $x = a$. At $t = 0$ the right wall is yanked *instantaneously* out to $x = 3a$. Because the move is instantaneous, the wave function has no time to change: an instant later it is still the old ground state,

$$\Psi(x, 0) = \sqrt{\frac{2}{a}} \sin\left(\frac{\pi x}{a}\right) \quad (0 \leq x \leq a), \quad \Psi(x, 0) = 0 \quad (a < x \leq 3a),$$

but the stationary states are now those of the wider well, $\chi_n(x) = \sqrt{2/3a} \sin(n\pi x/3a)$, with energies $E_n^{\text{new}} = n^2\pi^2\hbar^2/18ma^2$. An energy measurement is then made.

- What is the probability of finding the particle in the ground state of the new well?
- What is the probability of finding it in the first excited state of the new well?
- Suppose instead the particle had started in the *first excited* state of the old well. What is the probability of finding it in the first excited state of the new well?
- Which new state is the single most likely outcome, and what is its energy? Comment on the result.

a) For $n = 1$, $\int_0^a \sin(\frac{\pi x}{3a}) \sin(\frac{\pi x}{a}) dx = \frac{1}{2} \int_0^a [\cos \frac{2\pi x}{3a} - \cos \frac{4\pi x}{3a}] dx = \frac{9\sqrt{3}a}{16\pi}$, so $c_1 = \frac{2}{a\sqrt{3}} \cdot \frac{9\sqrt{3}a}{16\pi} = \frac{9}{8\pi}$ and

$$P_1 = \left(\frac{9}{8\pi}\right)^2 = \frac{81}{64\pi^2} \approx 0.128.$$

b) For $n = 2$, $\int_0^a \sin(\frac{2\pi x}{3a}) \sin(\frac{\pi x}{a}) dx = \frac{9\sqrt{3}a}{10\pi}$, so $c_2 = \frac{9}{5\pi}$ and

$$P_2 = \frac{81}{25\pi^2} \approx 0.328.$$

c) Now the initial state is $\sqrt{2/a} \sin(2\pi x/a)$ and we want $n = 2$ of the new well:

$$c = \frac{2}{a\sqrt{3}} \int_0^a \sin\left(\frac{2\pi x}{3a}\right) \sin\left(\frac{2\pi x}{a}\right) dx = -\frac{9}{16\pi}, \quad P = \frac{81}{256\pi^2} \approx 0.032.$$

d) The new $n = 3$ state $\chi_3 = \sqrt{2/3a} \sin(\pi x/a)$ is identical in shape to the old ground state on $[0, a]$, so $c_3 = \int_0^a \sqrt{2/3a} \sqrt{2/a} \sin^2(\pi x/a) dx = 1/\sqrt{3}$ and $P_3 = 1/3$, the largest single probability. Its energy is $E_3^{\text{new}} = 9\pi^2\hbar^2/18ma^2 = \pi^2\hbar^2/2ma^2 = E_1^{\text{old}}$: the most likely outcome has *exactly the original energy*.

Exercise 7. A particle of mass m is in the harmonic oscillator potential $V(x) = \frac{1}{2}m\omega^2 x^2$.

a) Show that $\langle x \rangle = 0$ in all stationary states ψ_n .

b) Show that $x^2\psi_0 = \frac{\hbar}{2m\omega} (\psi_0 + \sqrt{2}\psi_2)$, and find $\langle x^2 \rangle$ in the ground state.

c) Find $\langle x^4 \rangle$ in the ground state, and show that it equals $3 \langle x^2 \rangle^2$.

d) Find $\langle V \rangle$ and the standard deviation σ_V in the ground state.

a) $\hat{x}\psi_n = \sqrt{\frac{\hbar}{2m\omega}} (\sqrt{n+1}\psi_{n+1} + \sqrt{n}\psi_{n-1})$ is a combination of ψ_{n+1} and ψ_{n-1} , both orthogonal to ψ_n . Hence $\langle x \rangle = \int \psi_n \hat{x} \psi_n dx = 0$. (Equivalently, $|\psi_n|^2$ is even around the origin, so the odd integrand $x|\psi_n|^2$ integrates to zero.)

b) Expand $(\hat{a}_+ + \hat{a}_-)^2 = \hat{a}_+^2 + \hat{a}_+ \hat{a}_- + \hat{a}_- \hat{a}_+ + \hat{a}_-^2$ and act on ψ_0 . Since $\hat{a}_- \psi_0 = 0$, the $\hat{a}_-^2$ and $\hat{a}_+ \hat{a}_-$ terms (rightmost operator $\hat{a}_-$ hits ψ_0) vanish. The survivors are

$$\hat{a}_- \hat{a}_+ \psi_0 = \hat{a}_- \psi_1 = \psi_0, \quad \hat{a}_+^2 \psi_0 = \hat{a}_+ \psi_1 = \sqrt{2} \psi_2,$$

so $(\hat{a}_+ + \hat{a}_-)^2 \psi_0 = \psi_0 + \sqrt{2} \psi_2$ and

$$\hat{x}^2 \psi_0 = \frac{\hbar}{2m\omega} (\psi_0 + \sqrt{2} \psi_2).$$

Then $\langle x^2 \rangle = \int \psi_0 \hat{x}^2 \psi_0 dx = \frac{\hbar}{2m\omega} (1 + 0) = \boxed{\frac{\hbar}{2m\omega}}$ (the ψ_2 piece is orthogonal to ψ_0).

c) Because $\hat{x}^2$ is multiplication by the real function x^2 , one factor moves onto each copy of ψ_0 :

$$\langle x^4 \rangle = \int_{-\infty}^{\infty} \psi_0^* \hat{x}^4 \psi_0 dx = \int_{-\infty}^{\infty} (\hat{x}^2 \psi_0)^* (\hat{x}^2 \psi_0) dx = \int_{-\infty}^{\infty} |\hat{x}^2 \psi_0|^2 dx,$$

valid for any ψ_0 (the reality of ψ_0 is not needed here, only that x^2 is a real multiplier). From (b), $\hat{x}^2 \psi_0 = \frac{\hbar}{2m\omega}(\psi_0 + \sqrt{2} \psi_2)$, which is real, so the modulus-squared is an ordinary square and

$$\langle x^4 \rangle = \left(\frac{\hbar}{2m\omega} \right)^2 \int (\psi_0 + \sqrt{2} \psi_2)^2 dx = \left(\frac{\hbar}{2m\omega} \right)^2 (1 + 2) = \boxed{3 \left(\frac{\hbar}{2m\omega} \right)^2},$$

using orthonormality ($\int \psi_0^2 = \int \psi_2^2 = 1$, $\int \psi_0 \psi_2 = 0$). Since $\langle x^2 \rangle = \hbar/2m\omega$, this is $\langle x^4 \rangle = 3 \langle x^2 \rangle^2$, the Gaussian fourth-moment relation, as expected for the Gaussian ψ_0 . (The same step fails for $\hat{p}^4$, which differentiates rather than multiplies.)

d) The mean is $\langle V \rangle = \frac{1}{2} m \omega^2 \langle x^2 \rangle = \frac{1}{2} m \omega^2 \cdot \frac{\hbar}{2m\omega} = \boxed{\frac{1}{4} \hbar \omega}$ (half of E_0 , the virial result). For the spread,

$$\langle V^2 \rangle = \frac{1}{4} m^2 \omega^4 \langle x^4 \rangle = \frac{1}{4} m^2 \omega^4 \cdot 3 \left(\frac{\hbar}{2m\omega} \right)^2 = \frac{3 \hbar^2 \omega^2}{16},$$

so

$$\sigma_V^2 = \langle V^2 \rangle - \langle V \rangle^2 = \frac{3 \hbar^2 \omega^2}{16} - \frac{\hbar^2 \omega^2}{16} = \frac{\hbar^2 \omega^2}{8}, \quad \boxed{\sigma_V = \frac{\hbar \omega}{2\sqrt{2}}}.$$

(So $\sigma_V / \langle V \rangle = \sqrt{2}$: the spread in V exceeds its own mean.)