

Week 10 Worksheet

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Exercise 1. The matrix elements of x and p in the harmonic oscillator eigenbasis $\{|n\rangle\}$ are given by

$$x_{mn} = \langle m|x|n\rangle = \sqrt{\frac{\hbar}{2m\omega}}(\delta_{m,n+1}\sqrt{n+1} + \delta_{m,n-1}\sqrt{n})$$
$$p_{mn} = i\sqrt{\frac{m\omega\hbar}{2}}(\delta_{m,n+1}\sqrt{n+1} - \delta_{m,n-1}\sqrt{n}).$$

- a) For finite-dimensional matrices, show that $\text{tr}([A, B]) = 0$, where for an $(n \times n)$ -matrix A , the **trace** $\text{tr}(A)$ is the sum of the diagonal entries of A .

Hint: Write $\text{tr}(AB)$ in summation notation as $\sum_{i,j} c_{ij}$; that is, determine the scalars c_{ij} in terms of the matrix elements of A and B .

- b) *Paradox.* From the result of (a), it would seem that the expressions for the matrix elements of x and p imply that $\hbar = 0$, since $[x, p] = i\hbar$. Use the matrix elements x_{mn} and p_{mn} to calculate xp and px , and explain in detail why the paradoxical conclusion $\hbar = 0$ is invalid.

- c) **Bonus.** Derive the expressions for x_{mn} and p_{mn} given in the problem statement, using that

$$a^\dagger |n\rangle = \sqrt{n+1} |n+1\rangle$$
$$a |n\rangle = \sqrt{n} |n-1\rangle$$

and

$$a^\dagger = \frac{1}{\sqrt{2\hbar m\omega}}(m\omega x - ip).$$

Exercise 2. Unitary Operators. An operator U is called **unitary** if $UU^\dagger = U^\dagger U = 1$. This problem shows that in an infinite dimensional Hilbert space, if an operator preserves the norm of all vectors and possesses both a left and right inverse, then it is unitary.

- a) Let U be some operator on a finite-dimensional vector space, and let $|\psi'\rangle = U|\psi\rangle$. Show that $\langle\psi'|\psi'\rangle = \langle\psi|\psi\rangle$ if and only if U is unitary.
- b) Consider the Hilbert space for a particle moving in one dimension (which is infinite-dimensional). Let $|n\rangle$ be the usual harmonic oscillator eigenbasis, with $n = 0, 1, \dots$. Define an operator U by

$$U |n\rangle = |n+1\rangle.$$

Show that if $|\psi'\rangle = U |\psi\rangle$, then $\langle\psi'|\psi'\rangle = \langle\psi|\psi\rangle$. Then show that

$$U^\dagger U = \mathbb{1},$$

so that U has a left inverse, and it is $U^\dagger$. Does U have a right inverse? Is U unitary?

Hint: What is the value of $UU^\dagger$ on $|0\rangle$?

Exercise 3. *Some More Unitary Operators.* Let H be a hermitian operator and $U = e^{iH}$.

- a) Show that U is unitary. (Notice the analogy with scalars: If θ is real, then $e^{i\theta}$ has unit modulus.)
- b) If U_1, U_2 are unitary operators, show that $U_1 U_2$ is unitary. If $U_1 U_2$ is unitary, does that imply that U_1 and U_2 are?
- c) **Challenge.** Show that $\det(U) = e^{i \operatorname{tr}(H)}$.