Week 11 Worksheet Solutions

Jacob Erlikhman

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Exercise 1 (3D Harmonic Oscillator)**.** Consider 3-dimensional quantum mechanics with the potential $V(x,y,z) = \frac{1}{2}m\omega^2(x^2+y^2+z^2)$.

a) Show that $\psi = X(x)Y(y)Z(z)$ separates the problem into three 1D oscillators and give $E_{n_x n_y n_z}$ in terms of $N \equiv n_x + n_y + n_z$.

b) Give the energies and degeneracies of the three lowest levels, and a formula for the degeneracy $d(N)$.

c) Now make the oscillator *anisotropic*, $V = \frac{1}{2}m(\omega_x^2 x^2 + \omega_y^2 y^2 + \omega_z^2 z^2)$. Give the energies and compute $[\hat{H}, \hat{L}_z]$. For which frequencies does it vanish?

d) For $\omega_x = \omega_y = \frac{2}{3}\omega_z$, give the three lowest levels and their degeneracies.

a) $\hat{H} = \hat{H}_x + \hat{H}_y + \hat{H}_z$, three 1D oscillators, and the energies add: $E = \hbar\omega\big(N + \frac{3}{2}\big)$, $n_i = 0, 1, 2, \ldots$

b) $\frac{3}{2}\hbar\omega$ (1), $\frac{5}{2}\hbar\omega$ (3), $\frac{7}{2}\hbar\omega$ (6); in general, $d(N) = \binom{N+2}{2} = \frac{(N+1)(N+2)}{2}$.

c) $E = \hbar\big[\omega_x(n_x + \frac{1}{2}) + \omega_y(n_y + \frac{1}{2}) + \omega_z(n_z + \frac{1}{2})\big]$. The kinetic term commutes with $\hat{L}_z$ because it is rotationally symmetric. On the other hand, we can compute

$$[L_z, x^2] = [xp_y - yp_x, x^2] = -[yp_x, x^2] = -y[p_x, x^2] = 2i\hbar yx = -[L_z, y^2].$$

Thus,

$$[\hat{H}, \hat{L}_z] = -\tfrac{1}{2}m\big(\omega_x^2[\hat{L}_z, \hat{x}^2] + \omega_y^2[\hat{L}_z, \hat{y}^2]\big) = i\hbar m\big(\omega_y^2 - \omega_x^2\big)\hat{x}\hat{y},$$

which vanishes iff $\omega_x = \omega_y$. That makes sense: $\omega_x = \omega_y$ restores the rotational symmetry about the z-axis, so in that case L_z should be a symmetry of the system and must commute with the hamiltonian.

d) With $\omega_x = \omega_y \equiv \omega$, $\omega_z = \frac{3}{2}\omega$ and $E/\hbar\omega = n_x + n_y + \frac{3}{2}n_z + \frac{7}{4}$: $1.75\,\hbar\omega$ (1), $2.75\,\hbar\omega$ (2), $3.25\,\hbar\omega$ (1).

Exercise 2 (Infinite Spherical Well)**.** A particle of mass m is free inside a sphere of radius a and cannot escape. Use the radial equation from lecture,

$$-\frac{\hbar^2}{2m}\frac{d^2u}{dr^2} + \left[V(r) + \frac{\hbar^2}{2M}\frac{\ell(\ell+1)}{r^2}\right]u = Eu, \qquad u(r) = rR,$$

where $\int_0^\infty |u|^2 dr = 1$.

a) Set $\ell = 0$, write the equation and both boundary conditions, and name the 1D problem it coincides with.

b) Find the energies E_n and the normalized $R_n(r)$.

c) For $\ell = 1$ the regular solution is $u \propto r\, j_1(kr)$ with $j_1(x) = \sin(x)/x^2 - \cos(x)/x$. Find the first two lowest $\ell = 1$ energies (graphically—see the back of the worksheet).

No one got to this exercise, so it will be on the final review (and solutions will be posted along with those solutions).