Week 12 Worksheet

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Exercise 1. Consider the unitary operator $U(\hat{n}, \theta)$, for $\hat{n}$ a unit vector in 3-dimensions and θ a scalar, which acts on wavefunctions $\psi(\mathbf{x})$ by

$$U(\hat{n}, \theta)\psi(\mathbf{x}) = \psi(R(\hat{n}, \theta)^{-1}\mathbf{x}).$$

Here, $R(\hat{n}, \theta)$ is the rotation matrix in 3-dimensions by angle θ about $\hat{n}$. Since any unitary operator U can be written in the form $U = \exp(-iA/\hbar)$ for some operator A, Exercise 2 shows that $U(\hat{n}, \theta)$ can be written as $U(\hat{n}, \theta) = \exp(-i\theta\hat{n} \cdot \mathbf{K}/\hbar)$, for some vector of operators $\mathbf{K}$.

a) Suppose that θ is infinitesimal, and show that

$$U(\hat{n}, \theta)\psi(\mathbf{x}) = \psi(\mathbf{x}) - \theta(\hat{n} \times \mathbf{x}) \cdot \nabla\psi(\mathbf{x}).$$

Hint: Use a Taylor expansion of $\psi(\mathbf{x} - \boldsymbol{\varepsilon})$, where $\varepsilon = |\boldsymbol{\varepsilon}|$ is small.

b) Show that the result of (a) implies that $\mathbf{K} = \mathbf{x} \times \mathbf{p}$; hence, $\mathbf{K} = \mathbf{L}$ is angular momentum.
Hint: If $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are vectors, there is an identity

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b}).$$

a) We first need to check that the infinitesimal action of $R = R(\hat{n}, \theta)$ on a vector $\mathbf{x}$ is given by

$$R\mathbf{x} = \mathbf{x} + \theta\hat{n} \times \mathbf{x}, \tag{1}$$

where $\theta^2 = 0$. To see this, write $\mathbf{x} = \mathbf{x}_\parallel + \mathbf{x}_\perp$, where $\mathbf{x}_\parallel$ is parallel to $\hat{n}$ and $\mathbf{x}_\perp$ is perpendicular to it. Then, suppose that we choose our coordinate axes so that $\hat{n} = \hat{z}$; once we derive the coordinate independent presentation (1) in this case, we know that it works in all cases. In this basis, the rotation matrix R can be written

$$R = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Expanding this in small θ, we see that

$$R = \begin{bmatrix} 1 & -\theta & 0 \\ \theta & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Applying this to $\mathbf{x}_\perp = (x_1, x_2, 0)$, we find

$$\begin{aligned} R\mathbf{x}_\perp =& \begin{bmatrix} x_1 - \theta x_2 \\ \theta x_1 + x_2 \\ 0 \end{bmatrix} \\ =& \mathbf{x}_\perp + \theta \hat{n} \times \mathbf{x}, \end{aligned}$$

where we note that $\hat{n} \times \mathbf{x} = \hat{n} \times \mathbf{x}_\perp$. Thus,

$$R\mathbf{x} = \mathbf{x} + \theta \hat{n} \times \mathbf{x}.$$

Now, write

$$U(\hat{n}, \theta)\psi(\mathbf{x}) = \psi(R^{-1}\mathbf{x}) = \psi(\mathbf{x} - \theta \hat{n} \times \mathbf{x}) = \psi(\mathbf{x}) - \theta(\hat{n} \times \mathbf{x}) \cdot \nabla\psi(\mathbf{x})$$

as required.

b) We expand

$$\begin{aligned} U(\hat{n}, \theta) =& \exp(-i\theta \hat{n} \cdot \mathbf{K}/\hbar) \\ =& \mathbb{1} - \frac{i}{\hbar}\theta \hat{n} \cdot \mathbf{K}. \end{aligned}$$

On the other hand,

$$U(\hat{n}, \theta)\psi(\mathbf{x}) = \psi(\mathbf{x}) - \theta \hat{n} \cdot (\mathbf{x} \times \nabla\psi(\mathbf{x}).$$

Equating these two, we see that

$$\begin{aligned} \mathbf{K} =& -i\hbar \mathbf{x} \times \nabla \\ =& \mathbf{x} \times \mathbf{p} \\ =& \mathbf{L}. \end{aligned}$$

Exercise 2. For R a rotation matrix in 3-dimensions, let $U(R)$ be the operator defined by $U(R)\,|\mathbf{x}\rangle = |R\mathbf{x}\rangle$.

a) Show that $U = U(R)$ is unitary.
Hint: First, show that $\langle \mathbf{x}'|U^\dagger U|\mathbf{x}\rangle = \langle \mathbf{x}'|\mathbf{x}\rangle$ for all $\mathbf{x}$ and $\mathbf{x}'$. What else is left to show that $U(R)$ is unitary?

b) Show that $U(R_1)U(R_2) = U(R_1 R_2)$.

c) Show that $U(I) = \mathbb{1}$, where I is the identity (3×3)-matrix.

d) Show that $U(R^{-1}) = U(R)^{-1}$.

e) Show that if

$$|\psi'\rangle = U(R)\,|\psi\rangle\,,$$

then

$$\psi'(\mathbf{x}) = \psi(R^{-1}\mathbf{x}).$$

Hint: Use a resolution of the identity.

a) We need to show that $UU^\dagger = U^\dagger U = \mathbb{1}$. To do this, it is sufficient to identify all the matrix elements of the three operators. Following the hint, we check

$$\langle \mathbf{x}'|U^\dagger U|\mathbf{x}\rangle = \langle U\mathbf{x}'|U|\mathbf{x}\rangle = \langle R\mathbf{x}'|R\mathbf{x}\rangle = \delta(R(\mathbf{x}-\mathbf{x}')) = \delta(\mathbf{x}-\mathbf{x}') = \langle \mathbf{x}'|\mathbf{x}\rangle\,,$$

since $R(\mathbf{x}-\mathbf{x}') = 0$ if and only if $\mathbf{x}-\mathbf{x}' = 0$. To figure out what happens for $UU^\dagger$, we need to figure out what $U^\dagger$ does to kets $|\mathbf{x}\rangle$. To see this, note that

$$\langle \mathbf{x}'|U^\dagger|\mathbf{x}\rangle = \langle U\mathbf{x}'|\mathbf{x}\rangle = \langle R\mathbf{x}'|\mathbf{x}\rangle = \delta(R\mathbf{x}'-\mathbf{x}).$$

Now, this δ-function will vanish unless $\mathbf{x}' = R^{-1}\mathbf{x}$, so it is the same as

$$\delta(R(\mathbf{x}'-R^{-1}\mathbf{x})) = \delta(\mathbf{x}'-R^{-1}\mathbf{x}) = \langle \mathbf{x}'|R^{-1}\mathbf{x}\rangle\,.$$

Since this is true for all $\mathbf{x}'$, this gives that

$$U^\dagger|\mathbf{x}\rangle = |R^{-1}\mathbf{x}\rangle\,.$$

Now, we can calculate

$$\langle \mathbf{x}'|UU^\dagger|\mathbf{x}\rangle = \langle R^{-1}\mathbf{x}'|R^{-1}\mathbf{x}\rangle = \langle \mathbf{x}'|\mathbf{x}\rangle$$

since we can replace $R \leftrightarrow R^{-1}$ in the argument for $U^\dagger U$.

b) It is sufficient to check that the two operators in question match up on all kets $|\mathbf{x}\rangle$. We check

$$U(R_1)U(R_2)|\mathbf{x}\rangle = U(R_1)|R_2\mathbf{x}\rangle = |R_1R_2\mathbf{x}\rangle = U(R_1R_2)\mathbf{x}.$$

c) We check

$$U(I)|\mathbf{x}\rangle = |\mathbf{x}\rangle = \mathbb{1}|\mathbf{x}\rangle\,.$$

d) This follows from the answer to part (a), where we found $U^\dagger = U^{-1} = U(R^{-1})$.

e) Following the hint, we use a resolution of the identity to compute

$$\begin{aligned}|\psi'\rangle &= \int d^3x' U(R)|\mathbf{x}'\rangle\langle\mathbf{x}'|\psi\rangle \\ &= \int d^3x'\,|R\mathbf{x}'\rangle\langle\mathbf{x}'|\psi\rangle\,.\end{aligned}$$

Since we're interested in $\psi'(\mathbf{x}) = \langle\mathbf{x}|\psi'\rangle$, we should apply $\langle\mathbf{x}|$ to the above formula. This gives

$$\begin{aligned}\psi'(\mathbf{x}) &= \int d^3x'\,\langle\mathbf{x}|R\mathbf{x}'\rangle\,\psi(\mathbf{x}') \\ &= \int d^3x'\delta(\mathbf{x}-R\mathbf{x}')\psi(\mathbf{x}') \\ &= \psi(R^{-1}\mathbf{x}'),\end{aligned}$$

since $\mathbf{x} = R\mathbf{x}'$ implies $\mathbf{x}' = R^{-1}\mathbf{x}$.