Week 12 Worksheet

Jacob Erlikhman

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Exercise 1. Consider the unitary operator $U(\hat{n}, \theta)$, for $\hat{n}$ a unit vector in 3-dimensions and θ a scalar, which acts on wavefunctions $\psi(\mathbf{x})$ by

$$U(\hat{n}, \theta)\psi(\mathbf{x}) = \psi(R(\hat{n}, \theta)^{-1}\mathbf{x}).$$

Here, $R(\hat{n}, \theta)$ is the rotation matrix in 3-dimensions by angle θ about $\hat{n}$. Since any unitary operator U can be written in the form $U = \exp(-iA/\hbar)$ for some operator A, Exercise 2 shows that $U(\hat{n}, \theta)$ can be written as $U(\hat{n}, \theta) = \exp(-i\theta\hat{n} \cdot \mathbf{K}/\hbar)$, for some vector of operators $\mathbf{K}$.

a) Suppose that θ is infinitesimal, and show that

$$U(\hat{n}, \theta)\psi(\mathbf{x}) = \psi(\mathbf{x}) - \theta(\hat{n} \times \mathbf{x}) \cdot \nabla\psi(\mathbf{x}).$$

Hint: Use a Taylor expansion of $\psi(\mathbf{x} - \boldsymbol{\varepsilon})$, where $\varepsilon = |\boldsymbol{\varepsilon}|$ is small.

b) Show that the result of (a) implies that $\mathbf{K} = \mathbf{x} \times \mathbf{p}$; hence, $\mathbf{K} = \mathbf{L}$ is angular momentum.
Hint: If $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are vectors, there is an identity

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b}).$$

Exercise 2. For R a rotation matrix in 3-dimensions, let $U(R)$ be the operator defined by $U(R)\,|\mathbf{x}\rangle = |R\mathbf{x}\rangle$.

a) Show that $U = U(R)$ is unitary.
Hint: First, show that $\langle \mathbf{x}'|U^\dagger U|\mathbf{x}\rangle = \langle \mathbf{x}'|\mathbf{x}\rangle$ for all $\mathbf{x}$ and $\mathbf{x}'$. What else is left to show that $U(R)$ is unitary?

b) Show that $U(R_1)U(R_2) = U(R_1R_2)$.

c) Show that $U(I) = \mathbb{1}$, where I is the identity (3×3)-matrix.

d) Show that $U(R^{-1}) = U(R)^{-1}$.

e) Show that if

$$|\psi'\rangle = U(R)\,|\psi\rangle\,,$$

then

$$\psi'(\mathbf{x}) = \psi(R^{-1}\mathbf{x}).$$

Hint: Use a resolution of the identity.