Week 3 Worksheet Solutions

Jacob Erlikhman

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Exercise 1. Translation. Let $\psi(x)$ be the position space wavefunction of a particle, and let $\varphi(p)$ be its wavefunction in momentum space.

a) Find the momentum space wavefunction of the position wavefunction $\psi(x+a)$ for some constant a. Make sure your answer is in terms of $\varphi(p)$.

b) Write down the Taylor series expansion of $\psi(x+a)$ about $a=0$, and show it's the same as

$$e^{ipa/\hbar}\psi(x).$$

Remark. We say that $e^{-ipa/\hbar}$ is the **translation operator** (by a).

a) We do the Fourier transform to find

$$\frac{1}{\sqrt{2\pi\hbar}}\int e^{-ipx/\hbar}\psi(x+a)dx = \frac{1}{\sqrt{2\pi\hbar}}\int e^{-ip(x-a)}\psi(x)dx = e^{ipa/\hbar}\varphi(p).$$

b) This is easiest if you imagine that the function $\psi(x+a) = f_x(a)$ is a function of a at fixed x. We know how to expand these about $a=0$: The Taylor series expansion about $a=0$ is by definition

$$\begin{aligned} f_x(a) &= \sum_{n=0}^{\infty}\frac{f_x^{(n)}(0)}{n!}a^n \\ &= \sum \frac{a^n}{n!}\frac{\partial^n\psi(x)}{\partial x^n}. \end{aligned}$$

On the other hand, expanding

$$e^{ipa/\hbar}\psi(x) = \exp(a\partial_x)\psi(x)$$

as a power series, we find

$$\sum_{n=0}^{\infty}\frac{a^n}{n!}\partial_x^n\psi(x),$$

which exactly matches the expansion for $\psi(x+a)$.

Exercise 2. Probability Current. In class, you saw the quantity

$$J(x,t) = \frac{1}{2m}[\Psi^*(x,t)p\Psi(x,t) - \Psi(x,t)p\Psi^*(x,t)],$$

which is called **the probability current**.

a) Show, using the time-dependent Schrödinger equation, that

$$\frac{\partial J}{\partial x} + \frac{\partial \rho}{\partial t} = 0,$$

where $\rho(x,t) = |\Psi(x,t)|^2$ is the probability density. This equation is called the **continuity equation**.

b) Comment physically on what the continuity equation implies.
Hints: The 3D analog of the continuity equation is $\boldsymbol{\nabla} \cdot \mathbf{J} + \partial\rho/\partial t = 0$. If you have a charged particle of charge q, and you multiply the continuity equation by q, what do you get?

a) We first evaluate

$$\partial_t |\Psi|^2 = \partial_t \Psi^* \Psi + \Psi^* \partial_t \Psi = \left(-i\frac{\hbar}{2m}\partial_x^2 \Psi^* + \frac{i}{\hbar} V\Psi^*\right)\Psi + \Psi^*\left(i\frac{\hbar}{2m}\partial_x^2\Psi - \frac{i}{\hbar}V\Psi\right) =$$
$$= i\frac{\hbar}{2m}(\Psi^*\partial_x^2\Psi - \Psi\partial_x^2\Psi^*).$$

On the other hand,

$$\partial_x J = -i\frac{\hbar}{2m}[\Psi^*\partial_x^2\Psi - \Psi\partial_x^2\Psi^*],$$

where the terms of the form $|\partial_x \Psi|^2$ cancel. Once we have $\partial_t \rho$ and $\partial_x J$ in the above forms, we see that they exactly cancel.

b) By analogy with electromagnetism, we see that if we multiply the continuity equation by q we get the continuity equation for electromagnetism, which says that charge cannot be created or destroyed. So in QM the continuity equation must say that *probability* cannot be created or destroyed.