Week 4 Worksheet Solutions

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September 16, 2026

Exercise 1. Warm up. For these problems, you don't need to give rigorous proofs or all the details. Just make sure you understand the concepts.

a) Starting from the time-dependent Schrödinger equation (TDSE)

$$i\hbar\frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2 \Psi}{\partial x^2} + V(x)\Psi,$$

explain how to obtain the time-independent Schrödinger equation (TISE) $\hat{H}\psi = E\psi$. You don't have to go through the details of the derivation, but you do have to explain all of the steps.

b) Why is a general solution to the TISE a linear combination of definite energy solutions?
Hint: If $D\psi = 0$ is a differential equation and D is a *linear* differential operator, so that $D(a\psi_1 + b\psi_2) = aD\psi_1 + bD\psi_2$, then what does that tell you about linear combinations of solutions?

c) If we write down a linear combination $\psi(x) = \sum_i c_i\psi_i(x)$ as in (b), why is its time dependence given by $\Psi(x,t) = \sum_i c_i\Psi_i(x,t)$, where

$$\Psi_i(x,t) = e^{-iE_i t/\hbar}\psi_i(x)$$

is the solution to the TDSE given by the stationary state ψ_i with energy E_i?

a) This is explained in Griffiths.

b) You can check for yourself that $H - E = D$ is a linear operator. Thus, solutions to the TISE with energy E are linear combinations of such solutions.

c) Same as (b), but instead we check that the operator $H - i\hbar\partial/\partial t = D$ is linear. Then we know that solutions to the TDSE are just linear combinations of stationary state solutions.

Exercise 2. Degeneracy. In this exercise, you will show that same-energy solutions to the TISE that vanish anywhere on the x-axis, including $\pm\infty$, can never be degenerate. Suppose we have two solutions ψ, φ to the time-independent Schrödinger equation with *the same* energy E.

a) Show that in this case

$$W = \varphi\psi' - \psi\varphi'$$

is a constant (independent of x). W is called the **wronskian** of the two solutions.
Hint: Show that $dW/dx = 0$.

b) Suppose that ψ and φ both vanish at some point x, including the possibilities $x \to \infty$ or $x \to -\infty$. Show that in this case

$$\psi(x) = c\varphi(x)$$

for some constant c.

a) Following the hint, we want to show $dW/dx = 0$. First, we compute this as

$$\frac{dW}{dx} = \varphi\psi'' - \psi\varphi'',$$

since the cross terms cancel. Since we have double derivatives of ψ and φ appearing, we can obtain a term like $\varphi\psi''$ by multiplying the TISE for ψ by φ, and similarly for the term $\psi\varphi''$. Then, we subtract the two equations from each other to obtain

$$-\frac{\hbar^2}{2m}\frac{dW}{dx} = 0,$$

since the potentials and energies E for both ψ and φ are the same. Thus, W is a constant.

b) If ψ and φ both vanish somewhere, then we find that W is in fact 0. Thus, we have a differential equation

$$\frac{\psi'}{\psi} = \frac{\varphi'}{\varphi}.$$

We can solve this equation to find $\ln(\psi) = \ln(\varphi) + \ln(c)$ to obtain the relation

$$\psi(x) = c\varphi(x).$$

Bonus Exercise. Consider a free particle solution to the TISE (a free particle means $V(x) \equiv 0$), and write a stationary state as

$$\psi_k(x) = \frac{1}{\sqrt{2\pi\hbar}}e^{ikx},$$

where $p = \hbar k$.

a) What is the energy E_k of the stationary state ψ_k?

b) Is ψ_k normalizable?

c) Is the most general free particle solution to the TISE a linear combination of stationary states, or something more complicated?
Hint: Recall that any function on a bounded interval, like $[0, 2\pi]$, (equivalently, any *periodic* function) can be expressed as a Fourier *series*. If the interval isn't bounded, then any function can still be expressed in this way, but now as a Fourier *integral*.

a) Since $p^2/2m = E$, we have

$$E_k = \frac{\hbar^2 k^2}{2m}.$$

b) ψ_k isn't normalizable. In fact, its integral over the real line is undefined!

c) The most general solution is a Fourier integral

$$\psi(x) = \frac{1}{\sqrt{2\pi\hbar}} \int c_k e^{ikx} dx,$$

where the c_k are coefficients. In fact, the c_k assemble to give precisely the *function* $\varphi(k) = c_k$, where $\varphi(k)$ is the Fourier transform of $\psi(x)$.