Midterm 1 Review Solutions

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Exercise 1. Show that if $E < V(x)$ for all x, then $\psi(x) \equiv 0$.

We check

$$\begin{aligned} E &= \langle H \rangle \\ &= \langle \frac{p^2}{2m} + V(x) \rangle \\ &= \frac{1}{2m} \langle p^2 \rangle + \langle V \rangle \\ &= \frac{1}{2m} \langle p\psi | p\psi \rangle + \langle V \rangle . \end{aligned}$$

The first term is nonnegative, because it is proportional to $\int |p\psi|^2 dx$. On the other hand, the second term cannot be less than the minimum value of $V(x)$, which means neither can E.

Exercise 2. Suppose a free particle moves not on the x-axis but on a circle of radius R.

a) Find the stationary states and the allowed energies.

b) You should have found in (a) that there are degenerate states (i.e. linearly independent stationary states with the same energy). How is that consistent with the result of Exercise 2 of the second week 2 discussion section worksheet?

a) We need to solve the TISE with the periodic boundary conditions $\psi(0) = \psi(2\pi R)$. Set $2\pi R = L$. Then the free particle TISE is just

$$-\frac{\hbar^2}{2m}\psi'' = E\psi,$$

which has solutions

$$\psi(x) = Ae^{ikx} + Be^{-ikx}.$$

Enforcing the periodic boundary conditions, we find

$$A + B = Ae^{ikL} + Be^{-ikL}.$$

Similarly, the equation for the first derivative gives

$$A - B = Ae^{ikL} - Be^{-ikL}.$$

Adding these two equations together, we see that

$$e^{ikL} = 1,$$

so that $k = 2n\pi/L$. Unlike in the case of the free particle on the real line, these solutions *are* normalizable. Indeed, we see that we get two linearly independent solutions given by "right-moving" waves

$$\psi_+(x) = \frac{1}{\sqrt{L}} e^{2in\pi x/L}$$

and "left-moving" waves

$$\psi_-(x) = \frac{1}{\sqrt{L}} e^{-2in\pi x/L}.$$

b) In order to use the result of the cited exercise, we need to check that $\psi_-(x) = \psi_+(x) = 0$ for some value of x, i.e. we need to check that

$$e^{i\alpha x} = e^{-i\alpha x} = 0$$

for some x. But exponentials are never 0, so this is impossible. Hence, there is no contradiction.

Exercise 3. Calculate the momentum space wavefunctions for the harmonic oscillator, $\varphi_n(p)$.
Hint: Use the momentum space differential equation $H\varphi = E\varphi$ and that $x = i\hbar d/dp$.

We write down the TISE

$$\left(\frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2\right)\varphi = E\varphi.$$

Replacing $x = i\hbar d/dp$, we have the differential equation

$$\frac{p^2}{2m}\varphi - \frac{\hbar^2 m\omega^2}{2}\varphi'' = E\varphi.$$

Taking the hint from the ladder operators

$$a_\pm = \frac{1}{\sqrt{2\hbar m\omega}}(m\omega x \mp ip),$$

that if for the position space equation we want to change coordinates to

$$\xi = \sqrt{\frac{m\omega}{\hbar}}x,$$

then for the momentum space equation we want to change coordinates to

$$\eta = \frac{1}{\sqrt{m\omega\hbar}} p.$$

Making this substitution, we see that the equation becomes (note that the primes now represent derivatives with respect to η!)

$$\frac{\hbar\omega}{2}\eta^2\varphi - \frac{\hbar\omega}{2}\varphi'' = E\varphi,$$

which we can rewrite as

$$\varphi'' = (\eta^2 - K)\varphi,$$

where

$$K = \frac{2E}{\hbar\omega}$$

is a unitless constant. This has the immediate solution (copied from Griffiths Equation 2.86)

$$\varphi_n(p) = A_n \frac{1}{\sqrt{2^n n!}} H_n(\eta) e^{-\eta^2/2},$$

where A_n is a normalization. To determine A_n, it is sufficient to determine A_0, which will be given by (doing the gaussian integral $\int e^{-\alpha p^2} dp = \sqrt{\pi/\alpha}$)

$$\begin{aligned} A_0 =& (\pi m\omega\hbar)^{1/4} \\ =& A_n. \end{aligned}$$

Exercise 4. ***Griffiths 2.43.*** Solve the TISE for a centered infinite square well with a δ-funciton barrier in the middle:

$$V(x) = \begin{cases} \alpha\delta(x), -a < x < a \\ \infty, |x| \geq a \end{cases}.$$

Don't bother to normalize your solutions, but find the allowed energies. Comment how the energies compare with the corresponding energies in the absence of the δ-function. Explain why the odd solutions are not affected by the δ-function. Comment on the limiting cases $\alpha \to 0$ and $\alpha \to \infty$.

It is convenient to analyze separately the even and odd solutions to the TISE. For example, the even solution is given by

$$\psi(x) = \begin{cases} A\sin(kx) + B\cos(kx), & x \in [0, a) \\ -A\sin(kx) + B\cos(kx), & x \in (-a, 0] \end{cases}.$$

This is already continuous at $x = 0$, so we check the equation for the discontinuity in ψ' :

$$-\frac{\hbar^2}{m} Ak + \alpha B = 0.$$

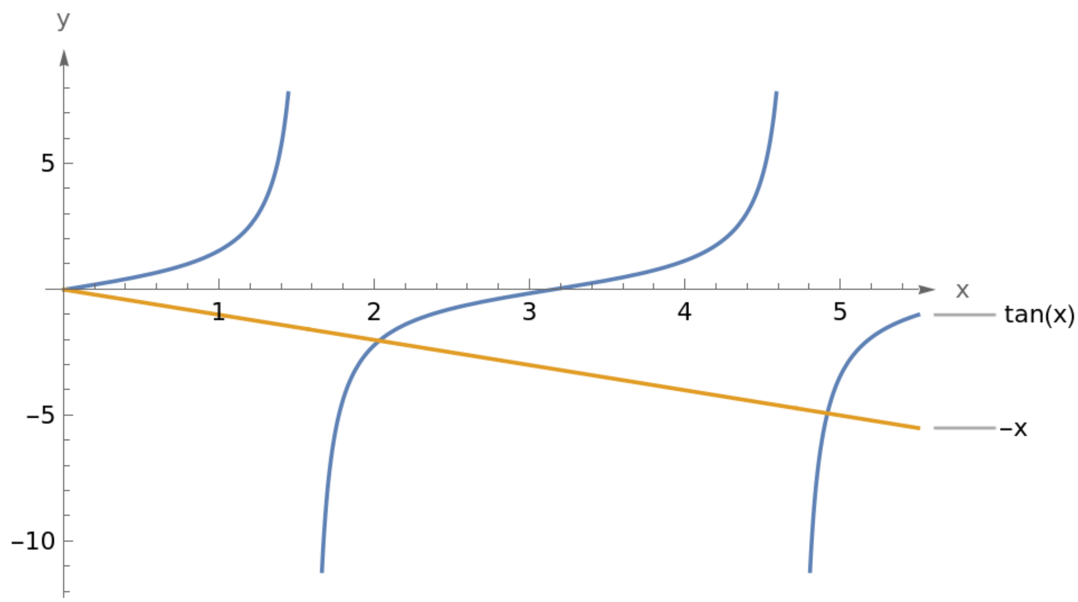
This has the solution

$$B = \frac{\hbar^2 k}{m\alpha} A.$$

We then impose the condition that $\psi(a) = 0$, which gives

$$\tan(ka) = -\frac{\hbar^2 k}{m\alpha},$$

a transcendental equation in k. To find the energies, we can solve this equation graphically (see the figure). Notice in the figure that as $\alpha \to 0$, the value of ka approaches $n\pi/2$, so that at $\alpha = 0$, the energies become the same energies as those of the infinite square well. For $\alpha > 0$, the energies are slightly higher than those of the infinite square well. As $\alpha \to \infty$, the intersections instead approach $n\pi$, so the allowed energies become those for the infinite square well of width a. This also makes sense, because at $\alpha \to \infty$, the δ-function barrier becomes impenetrable. For the odd solutions, we have



$$\psi(x) = \begin{cases} A\sin(kx) + B\cos(kx), & x \in [0, a) \\ A\sin(kx) - B\cos(kx), & x \in (-a, 0] \end{cases}.$$

Continuity at $x = 0$ gives that $B = 0$. The discontinuity equation becomes $0 = 0$, so it doesn't give us anything new. $\psi(a) = 0$ implies that

$$A\sin(ka) = 0,$$

so that $k = n\pi/2$. This means the energies are *the same* as those for the infinite square well. This makes sense, because the odd solutions are those with a node at 0; hence, they can't "feel" the δ-function.

Exercise 5. ***Griffiths 2.27-8.*** Consider the double δ-function potential

$$V(x) = -\alpha[\delta(x+a) + \delta(x-a)],$$

where $\alpha, a > 0$.

a) Sketch this potential.

b) How many bound states does it possess? Find the allowed energies for $\alpha = \hbar^2/ma$ and $\alpha = \hbar^2/4ma$, and sketch the wave functions.

c) What are the bound state energies in the limit cases (i) $a \to 0$ and (ii) $a \to \infty$ (holding α fixed)? Explain why your answers are reasonable.

d) Find the transmission coefficient.

a) See the handwritten supplement.

b) Since the potential is even, we can again look for even or odd solutions. The even solutions are

$$\psi(x) = \begin{cases} Ae^{\kappa x}, & x < -a \\ B\cosh(\kappa x), & x \in (-a, a) . \\ Ae^{-\kappa x}, & x > a \end{cases}$$

Continuity at a gives

$$B\cosh(\kappa a) = Ae^{-\kappa a}.$$

The discontinuity in ψ' at $x = a$ is given by

$$\kappa(-Ae^{-\kappa a} - B\sinh(\kappa a)) = -\frac{2m\alpha}{\hbar^2}\psi(a).$$

This has solution

$$\tanh(\kappa a) = \frac{2m\alpha}{\kappa\hbar^2} - 1.$$

We can solve this graphically, where we see that there is only one solution. If $\alpha = \hbar^2/2ma$, then we see that the solution is around $\kappa a = 0.62$ from the graph. Since

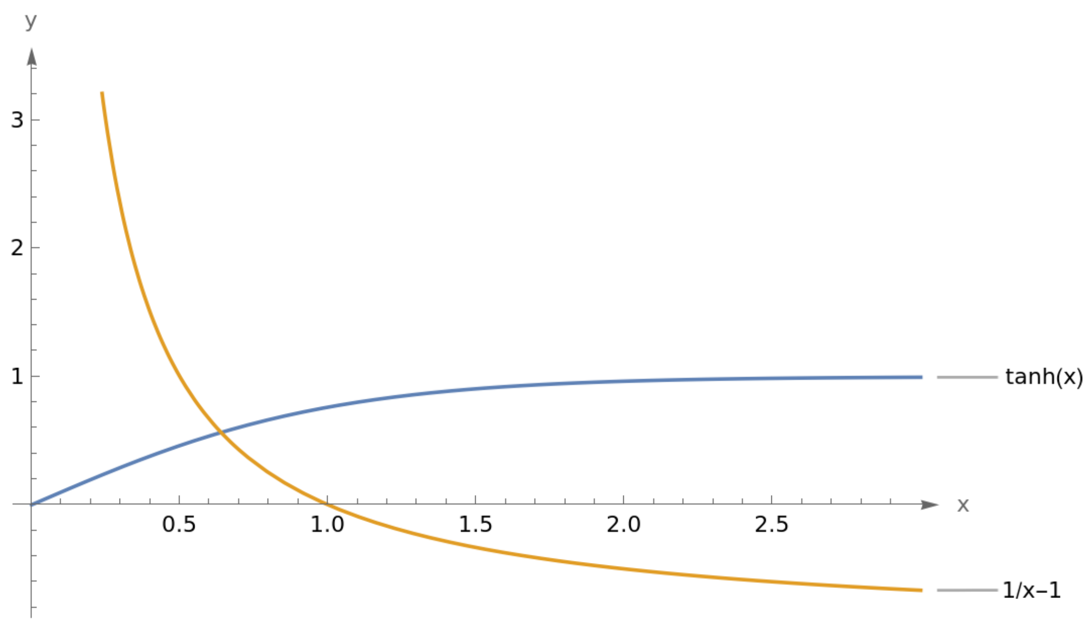
$$\kappa = \frac{\sqrt{-2mE}}{\hbar},$$

we see that

$$E = 0.62^2\frac{\hbar^2}{2ma^2}.$$

Now for the odd solutions, we have

$$\psi(x) = \begin{cases} -Ae^{\kappa x}, & x < -a \\ B\sinh(\kappa x), & x \in (-a, a) . \\ Ae^{-\kappa x}, & x > a \end{cases}$$



Continuity at a gives

$$Ae^{-\kappa a} = B\sinh(\kappa a).$$

The equation for the discontinuity in ψ' is

$$\kappa[-Ae^{-\kappa A} - B\cosh(\kappa a)] = \frac{-2m\alpha}{\hbar^2}\psi(a).$$

This has the solution

$$\coth(\kappa a) = \frac{2m\alpha}{\hbar^2\kappa} - 1.$$

Again, we can solve this graphically. Playing around with the value of $\xi > 0$ in the equation

$$\coth(x) = \frac{\xi}{x} - 1,$$

you will see that for $\xi < 1$ there is no solution, while for $\xi > 1$ there is a solution. This can be understood quantitatively as follows. First, we massage the above equation into the form (add 1 to $\coth(x)$ to get $2/(1-e^{-2x})$)

$$e^{-2x} = 1 - 2\eta x$$

for $\eta = 1/\xi$. The slope of the tangent at 0 to e^{-2x} is -2, while the slope of the tangent to $1-2\eta x$ at 0 is -2η. These will be the same precisely when $\eta = 1$. When $\eta > 1$, there will be no solution, as e^{-2x} will go to 0 more slowly than the straight line with slope 2η. On the other hand, when $\eta < 1$, these will meet somewhere. Thus, we see that there is one bound state if $\alpha \leq \hbar^2/2ma$ and two bound states if $\alpha > \hbar^2/2ma$, since $\eta = 1/\xi = \hbar^2/2m\alpha a$.

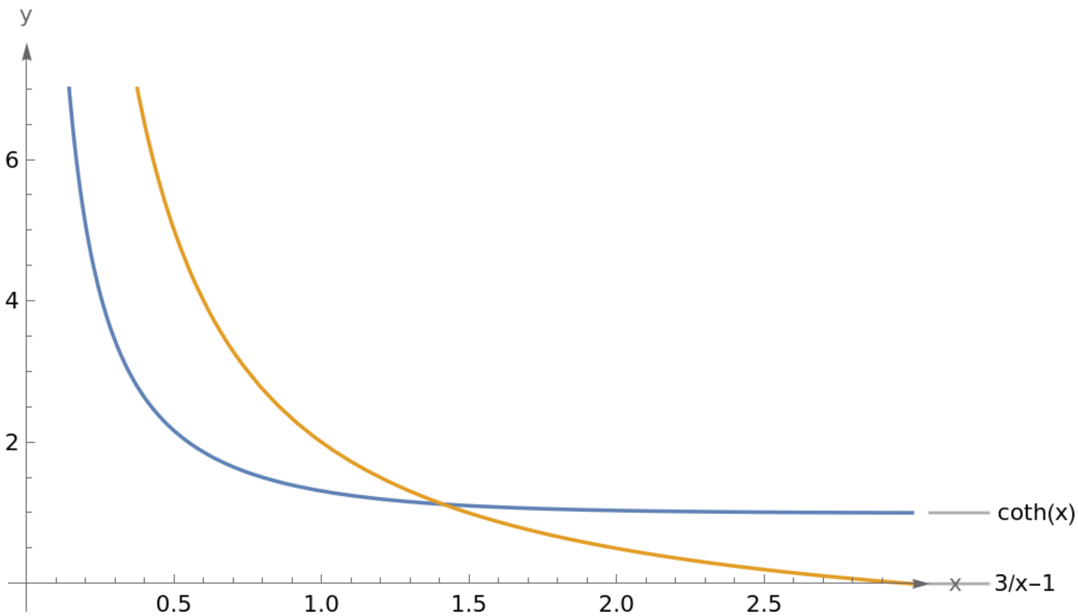


Figure 1: $\xi = 3 > 1$

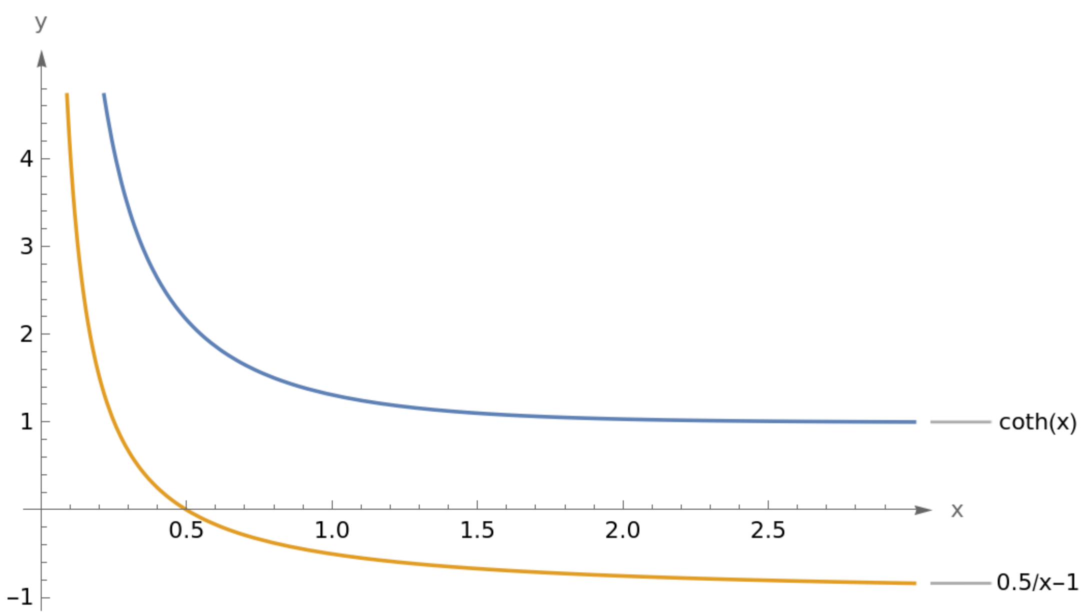


Figure 2: $\xi = 0.5 < 1$

c) As $a \to 0$, we see that we're in the setting of one bound state. This implies that $\kappa a \to 0$, so that $\tanh(\kappa a) \to 0$. This gives

$$\frac{2m\alpha}{\hbar^2} \approx \kappa.$$

We then have

$$\begin{aligned} E &= -\frac{\hbar^2\kappa^2}{2m} \\ &= -\frac{2m\alpha^2}{\hbar^2}, \end{aligned}$$

which exactly matches the energy for a single δ-function barrier given by $2\alpha\delta(x)$. This makes sense, because this is saying the two δ-functions have superposed to give a single one, twice as strong.

As $a \to \infty$, we get two bound states. In this case, $\tanh(\kappa a) \to 1$, so we get

$$\kappa_1 = \frac{m\alpha}{\hbar^2}.$$

This gives an energy of

$$E_1 = -\frac{m\alpha^2}{2\hbar^2}.$$

Similarly, $\coth(\kappa a) \to 1$, so this gives

$$E_2 = E_1.$$

So the two states become degenerate! If we take the sum of the wavefunctions which correspond to these, we would get an even solution, which will correspond to a state bound by the δ-function at $x \to \infty$. On the other hand, if we take the difference, we would get an odd solution, which corresponds to a state bound by the δ-function at $x \to -\infty$.

d) There are three regions with wavefunctions here, where in the region $x > a$ we only allow a right-moving wave:

$$\psi = \begin{cases} Ae^{ikx} + Be^{-ikx}, & x \in (-\infty, -a) \\ Ce^{ikx} + De^{-ikx}, & x \in (-a, a) \\ Fe^{ikx}, & x \in (a, \infty) \end{cases}.$$

To determine the transmission coefficient T, we need to determine A and F, since

$$T = \frac{|F|^2}{|A|^2}.$$

This is just an exercise in matching the boundary conditions. For the first boundary at $x = -a$, continuity gives

$$Ae^{-ika} + Be^{ika} = Ce^{-ika} + De^{ika}.$$

Continuity at $x = a$ gives

$$Ce^{ika} + De^{-ika} = Fe^{ika}.$$

The discontinuity in the first derivative at $x = -a$ gives

$$ik\left[Ce^{-ika} - De^{ika} - Ae^{-ika} + Be^{ika}\right] = \frac{2m\alpha}{\hbar^2}\left(Ce^{-ika} + De^{ika}\right).$$

Similarly, this discontinuity at $x = a$ is

$$ik\left[Fe^{ika} - Ce^{ika} + De^{-ika}\right] = \frac{2m\alpha}{\hbar^2}\left(Ce^{ika} + De^{-ika}\right).$$

Now, it's just a lot of algebra. Eventually, you should get to an equation that looks like

$$\frac{F}{A} = \frac{ik\hbar^2 e^{-2ika}}{m\alpha\left[2k + 2i\frac{m\alpha}{\hbar^2}\sin(2ka) + e^{2ika}\right]}.$$

At this point, we need to multiply the numerator and denominator by the complex conjugate of the denominator. This will reduce to

$$\frac{F}{A} = \frac{ik\hbar^2 e^{-2ika}\left[2k - 2i\frac{m\alpha}{\hbar^2}\sin(2ka) + e^{-2ika}\right]}{m\alpha\left[4k^2 + \frac{4m^2\alpha^2}{\hbar^2}\sin^2(2ka) + 1 + 4k\cos(ka) - \frac{4m\alpha}{\hbar^2}\sin^2(2ka)\right]}.$$

Finally, you can take the magnitude squared to get

$$\frac{|F|^2}{|A|^2} = \frac{k^2\hbar^4}{m^2\alpha^2\left[4k^2 + \frac{4m^2\alpha^2}{\hbar^2}\sin^2(2ka) + 1 + 4k\cos(ka) - \frac{4m\alpha}{\hbar^2}\sin^2(2ka)\right]}.$$

Exercise 6. ***Griffiths 2.33.***

a) Find the transmission coefficient for a rectangular barrier; that is, a potential with $V(x) = V_0 > 0$ in the region $x \in (-a, a)$. Treat separately the three cases $E < V_0$, $E = V_0$, and $E > V_0$.

b) Explain in what sense a plane wave solution $\psi(x) = Ae^{ikx}$ is a "wave traveling to the right." Same thing for plane waves $\psi(x) = Ae^{-ikx}$ as a "wave traveling to the left."

a) We start with $E < V_0$. As in the previous exercise, we have three regions:

$$\psi = \begin{cases} Ae^{ikx} + Be^{-ikx}, & x \in (-\infty, -a) \\ Ce^{\kappa x} + De^{-\kappa x}, & x \in (-a, a) \\ Fe^{ikx}, & x \in (a, \infty) \end{cases},$$

where

$$k = \frac{\sqrt{2mE}}{\hbar}$$

and

$$\kappa = \frac{\sqrt{2m(V_0 - E)}}{\hbar}.$$

Continuity at $x = -a$ gives

$$Ae^{-ika} + Be^{ika} = Ce^{-\kappa a} + De^{\kappa a}.$$

Continuity at $x = a$ gives

$$Ce^{\kappa a} + De^{-\kappa a} = Fe^{ika}.$$

Continuity of the first derivative at $x = -a$ gives

$$\kappa \left(Ce^{-\kappa a} - De^{\kappa a}\right) = ik\left(Ae^{-ika} - Be^{ika}\right).$$

Continuity of the first derivative at $x = a$ gives

$$ikFe^{ika} = \kappa\left(Ce^{\kappa a} - De^{-\kappa a}\right).$$

Again, it's now a lot of algebra. Eventually, you should get to an equation that gives

$$\frac{F}{A} = \frac{e^{-2ika}}{\cosh(2\kappa a) + i\frac{\kappa^2 - k^2}{2\kappa k}\sinh(2\kappa a)}.$$

Taking the modulus squared of this, we get

$$\frac{|F|^2}{|A|^2} = \frac{1}{\cosh^2(2\kappa a) + \frac{(\kappa^2 - k^2)^2}{4\kappa^2 k^2}\sinh^2(2\kappa a)}.$$

For $E > V_0$, this is the same as the formula in the book except with $V_0 \to -V_0$.

For $E = V_0$, we have the general form for ψ

$$\psi(x) = \begin{cases} Ae^{ikx} + Be^{-ikx}, & x \in (-\infty, -a) \\ C + Dx, & x \in (-a, a) \\ Fe^{ikx}, & x \in (a, \infty) \end{cases},$$

since the TISE becomes

$$\psi'' = 0$$

in the region $x \in (-a, a)$. Our four equations are now

$$\begin{aligned} Ae^{-ika} + Be^{ika} &= C - Da \\ C + Da &= Fe^{ika} \\ ik\left(Ae^{-ika} - Be^{ika}\right) &= D \\ D &= ikFe^{ika}. \end{aligned}$$

Less algebra than before gives

$$\frac{F}{A} = \frac{e^{-2ika}}{1 - ika}$$

which immediately gives

$$\frac{|F|^2}{|A|^2} = \frac{1}{1 + k^2 a^2}.$$

b) If we tack on the canonical time dependence $e^{-iEt/\hbar}$ to the wave e^{ikx}, we find

$$\Psi(x,t) = e^{ikx - ik^2\hbar t/2m}.$$

This is of the form

$$e^{i(kx - vt)},$$

which is precisely a wave traveling towards $x \to \infty$ (with speed v). If you haven't seen this before, notice that as t increases up from 0, we get a decrease in x. To "move this back to 0," we would need to make the change of variables $kx \to kx + vt$, so we see that kx has *increased*. Similarly, if we do the same to the wave e^{-ikx}, we get something of the form

$$e^{i(kx+vt)},$$

which is a wave traveling towards $x \to -\infty$.

Exercise 7. ***Griffiths 2.47*** Consider a double finite square well:

$$V(x) = \begin{cases} -V_0, & x \in (-a - b/2, b/2) \\ -V_0, & x \in (b/2, a + b/2) \\ 0, & \text{otherwise} \end{cases}.$$

a) Sketch this potential.

b) Sketch the allowed wave functions for the cases (i) $b = 0$, (ii) $b \approx a$, and (iii) $b \gg a$. Include also the ground state and first excited state.

c) Qualitatively, how do the corresponding energies of the ground and first excited states vary, as b goes from 0 to ∞? Sketch $E_1(b)$ and $E_2(b)$ on the same graph.
Hint: Recall the solutions to a single finite square well between $-a$ and a of depth V_0 are obtained from the equation

$$\tan\left(\frac{a}{\hbar}\sqrt{2m(E + V_0)}\right) = \sqrt{\frac{V_0}{E + V_0} - 1}.$$

What happens to the solutions to this equation when $z_0 = \frac{a}{\hbar}\sqrt{2mV_0}$ is large? It may help to first write the equation in terms of a unitless variable z and unitless constant z_0. Use this to make a qualitative statement about the behavior of the energies.

d) Suppose we use this potential as a 1-dimensional model for the potential energy experienced by an electron in a diatomic molecule. If the nuclei of the molecule are free to move, they will adopt the configuration of minimum energy. Use (b) to explain whether the electron pulls the nuclei together or pushes them apart.

The solution to this exercise is in the handwritten supplement.