Midterm 2 Review

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Exercise 1. We call an operator **normal** if it commutes with its hermitian conjugate, $[A, A^\dagger] = 0$. In particular, hermitian, anti-hermitian, and unitary operators are all normal, so this is a more general class of operators than hermitian operators.

a) Show that if A is normal and $A\,|u\rangle = a\,|u\rangle$ for some nonzero $|u\rangle$, then $A^\dagger\,|u\rangle = a^*\,|u\rangle$.

b) Show that (a) implies that the eigenbras of A are the hermitian conjugates of the eigenkets, and the left spectrum is identical to the right spectrum.

c) Show that the eigenspaces corresponding to distinct eigenvalues of a normal operator are orthogonal. This is a generalization of the proof for hermitian operators.

Exercise 2. *Unitary Operators.* An operator U is called **unitary** if $UU^\dagger = U^\dagger U = 1$. This problem shows that in an infinite dimensional Hilbert space, if an operator preserves the norm of all vectors and possesses both a left and right inverse, then it is unitary.

a) Let U be some operator on a finite-dimensional vector space, and let $|\psi'\rangle = U\,|\psi\rangle$. Show that $\langle\psi'|\psi'\rangle = \langle\psi|\psi\rangle$ if and only if U is unitary.

b) Consider the Hilbert space for a particle moving in one dimension (which is infinite-dimensional). Let $|n\rangle$ be the usual harmonic oscillator eigenbasis, with $n = 0, 1, \ldots$ Define an operator U by

$$U\,|n\rangle = |n+1\rangle\,.$$

Show that if $|\psi'\rangle = U\,|\psi\rangle$, then $\langle\psi'|\psi'\rangle = \langle\psi|\psi\rangle$. Then show that

$$U^\dagger U = \mathbb{1},$$

so that U has a left inverse, and it is $U^\dagger$. Does U have a right inverse? Is U unitary?

Exercise 3. *Some More Unitary Operators.* Let H be a hermitian operator and $U = e^{iH}$.

a) Show that U is unitary. (Notice the analogy with scalars: If θ is real, then $e^{i\theta}$ has unit modulus.)

b) If U_1, U_2 are unitary operators, show that U_1U_2 is unitary. If U_1U_2 is unitary, does that imply that U_1 and U_2 are?

c) ***Challenge.*** Show that $\det(U) = e^{i\,\mathrm{tr}(H)}$.

Exercise 4. *More Fun with Pauli Matrices*. Consider the 2×2 matrices,

$$\mathbb{1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

The matrices $\sigma_x, \sigma_y, \sigma_z$ are called the **Pauli matrices**; we will also denote them $\sigma_1, \sigma_2, \sigma_3$.

a) Show that the result from homework 5,

$$\sigma_i \sigma_j = \delta_{ij} + i\,\varepsilon_{ijk}\sigma_k, \tag{1}$$

implies that

$$[\sigma_i, \sigma_j] = 2i\,\varepsilon_{ijk}\sigma_k$$

and

$$\{\sigma_i, \sigma_j\} = \sigma_i\sigma_j + \sigma_j\sigma_i = 2\delta_{ij}.$$

b) What we mean by a **vector operator** is a vector *of* operators; for example, $\mathbf{A} = (A_x, A_y, A_z)$, where A_x, A_y, and A_z are operators. Two vector operators are said to commute if all of their components commute. Show that, in general, $\mathbf{A} \cdot \mathbf{B} \neq \mathbf{B} \cdot \mathbf{A}$ and $\mathbf{A} \times \mathbf{B} \neq -\mathbf{B} \times \mathbf{A}$.

c) Let $\hat{n}$ be an arbitrary unit vector and θ an arbitrary angle. Show that

$$\exp(-i\theta\boldsymbol{\sigma} \cdot \hat{n}) = \mathbb{1}\cos\theta - i(\boldsymbol{\sigma} \cdot \hat{n})\sin\theta.$$

We often drop the $\mathbb{1}$, it being understood that a scalar like $\cos\theta$ is multiplied by the identity matrix. *Hint*: The power series for sine and cosine will be useful here.

d) Show that if A and B are square matrices, then

$$\mathrm{tr}(AB) = \mathrm{tr}(BA).$$

e) Show that $\mathrm{tr}(\sigma_i) = 0$. Then, use Equation 1 to find a simple expression for $\mathrm{tr}(\sigma_i\sigma_j)$.

f) The set of four 2×2 matrices $\{\mathbb{1}, \boldsymbol{\sigma}\}$ forms a basis in the space of 2×2 matrices; that is, an arbitrary 2×2 matrix can be expressed as a linear combination of these four matrices. Thus, if M is some 2×2 matrix, we can always write

$$M = a\mathbb{1} + \mathbf{b} \cdot \boldsymbol{\sigma},$$

which follows from the fact that $\mathbb{1}$ and the Pauli matrices are linearly independent, as can be seen by just staring at the four matrices. Find nice expressions for the expansion coefficients a and $\mathbf{b} = (b_1, b_2, b_3)$. Do this by taking traces or by multiplying by a Pauli matrix and then taking traces. Show that M is hermitian if and only if a and b_i are real.

g) Express the following as linear combinations of the Pauli matrices and $\mathbb{1}$.

i) $(\mathbb{1} + i\sigma_x)^{1/2}$
Hint: Relate it to a certain half-rotation.

ii) $(2\mathbb{1} + \sigma_x)^{-1/2}$.

iii) σ_x^{-1}.

h) Suppose that M is only nonzero in the (r, s) index, where it has a 1. That is, suppose we can write

$$M_{ij} = \delta_{ir}\delta_{js}.$$

Use this to find a nice expression for $(\sigma_m)_{ij}(\sigma_m)_{kl}$, where we again use the convention that the m index is summed over, in terms of Kronecker deltas.

Exercise 5. *Even More Fun with Pauli Matrices.* In this exercise, you will give an explanation of the following fact from Week 5 Worksheet 2. Let $\hat{n}$ be a unit vector. By Exercise 2(a) of the same worksheet, $\hat{z}$ can be rotated into $\hat{n}$ using just two angles. Let R be the (3×3)-matrix which rotates $\hat{z}$ into $\hat{n}$, i.e. $R\hat{z} = \hat{n}$. Now, let $U = e^{-\frac{i}{2}\alpha\sigma_z}e^{-\frac{i}{2}\beta\sigma_y}$, then

$$U^\dagger \boldsymbol{\sigma} U = R\boldsymbol{\sigma},$$

where on the RHS we understand R as rotating the components of the *vector* $\boldsymbol{\sigma}$, just as it would rotate the components of a vector $\mathbf{a}$ whose entries were scalars.

a) Use Exercise 4(c) to calculate $U^\dagger \boldsymbol{\sigma} U$.

b) Prove the identities

$$[\hat{n} \cdot \boldsymbol{\sigma}, \boldsymbol{\sigma}] = -2i\hat{n} \times \boldsymbol{\sigma}, \qquad (\hat{n} \cdot \boldsymbol{\sigma})\boldsymbol{\sigma}(\hat{n} \cdot \boldsymbol{\sigma}) = 2\hat{n}(\hat{n} \cdot \boldsymbol{\sigma}) - \boldsymbol{\sigma}.$$

c) Use (b) to rewrite your result from (a) as

$$U^\dagger \boldsymbol{\sigma} U = \cos(\theta)\boldsymbol{\sigma} + (1 - \cos\theta)\hat{n}(\hat{n} \cdot \boldsymbol{\sigma}) + \sin(\theta)\hat{n} \times \boldsymbol{\sigma}.$$

d) ***Challenge.*** Give a physical explanation for why the result of (c) is reasonable for rotations, and why it should be the same as $R\boldsymbol{\sigma}$.

Exercise 6. Ignore the fact that the hydrogen atom is a three-dimensional problem, and imagine that

$$H = \frac{p^2}{2m} - \frac{e^2}{r}$$

corresponds to a one-dimensional problem, where $p^2 = p_x^2 + p_y^2 + p_z^2$ and $r^2 = x^2 + y^2 + z^2$. Assuming that

$$\Delta p \Delta r \geq \hbar/2,$$

estimate the ground state energy. (The actual ground state energy is about -13.6 eV).
Hint: To make your calculation easier, recall that $mc^2 = 511$ keV is the electron rest energy. You can use that the Bohr radius is

$$\begin{aligned} a_0 &= \frac{\hbar^2}{me^2} \\ &\approx 137 \cdot \frac{\hbar}{mc}. \end{aligned}$$