Final Review

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Exercise 1.

a) Do problem 4.59 from Griffiths: If two electrons are in the spin singlet state, $S_a^{(1)}$ is the component of spin angular momentum of particle 1 along the axis defined by a vector **a**, and $S_b^{(2)}$ is the spin angular momentum of particle 2 along the axis defined by some other vector **b**, show that

$$\left\langle S_a^{(1)} S_b^{(2)} \right\rangle = -\frac{\hbar^2}{4} \hat{a} \cdot \hat{b}.$$

b) Redo problem 4.59 from Griffiths by using Exercise 4(c) from the midterm review: If two electrons are in the spin singlet state, $S_z^{(1)}$ is the component of spin angular momentum of particle 1 along the z-axis, and $S_\theta^{(2)}$ is the spin angular momentum of particle 2 along the $\hat{r} = (\theta, 0)$ axis, show that

$$\left\langle S_z^{(1)} S_\theta^{(2)} \right\rangle = -\frac{\hbar^2}{4} \cos\theta.$$

Exercise 2. *Hidden Symmetries.* In classical mechanics, $\frac{1}{r}$ potentials have an additional conserved quantity that is rarely covered in introductory courses. This quantity is called the **Runge-Lenz vector**, and it is given by

$$\mathbf{F} = \frac{1}{m} \mathbf{p} \times \mathbf{L} - \frac{\gamma}{r} \mathbf{r},$$

where γ is the constant associated to the potential $V(r) = -\gamma/r$, e.g. $\gamma = e^2/4\pi\varepsilon_0$ or $\gamma = MG$.

a) If we replace all the classical dynamical variables in the above expression by quantum operators, explain why the result is ambiguous.
Hints: When we upgrade $\mathbf{L} = \mathbf{r} \times \mathbf{p}$ to a quantum operator, note that $\mathbf{L} = -\mathbf{p} \times \mathbf{r}$ as operators. Why? Does $\mathbf{p} \times \mathbf{L} = -\mathbf{L} \times \mathbf{p}$ as operators?

b) It turns out[1] that the correct quantum mechanical version of **F** is

$$\mathbf{F} = \frac{1}{2m} (\mathbf{p} \times \mathbf{L} - \mathbf{L} \times \mathbf{p}) - \frac{\gamma}{r} \mathbf{r}.$$

Show that for the hydrogen atom hamiltonian $H = p^2/2m - \gamma/r$, $[H, \mathbf{F}] = \mathbf{0}$, so that **F** is a symmetry.

[1] The way you would prove this is by matching Poisson bracket relations with **F** in classical mechanics to corresponding ones in quantum mechanics (upgrading the Poisson brackets to commutators). This would then allow you to determine the right combination of $\mathbf{p} \times \mathbf{L}$ and $\mathbf{L} \times \mathbf{p}$ to take.

Exercise 3.

a) Show that the average momentum vanishes, $\langle \mathbf{p} \rangle = \mathbf{0}$, in any bound state of the hamiltonian $H = p^2/2m + V(\mathbf{x})$.

b) The following alleged solution to (a) is given in a certain textbook.

Since $p_x = \frac{im}{\hbar}[H, x]$, it follows that

$$\langle p_x \rangle = \frac{im}{\hbar}(\langle \psi | H x | \psi \rangle - \langle \psi | x H | \psi \rangle).$$

Using $H \left| \psi \right\rangle = E \left| \psi \right\rangle$ and $\left\langle \psi \right| H = \left\langle \psi \right| E$, we obtain $\langle p_x \rangle = 0$, hence $\langle \mathbf{p} \rangle = \mathbf{0}$.

This argument, if valid, would imply that $\langle \mathbf{p} \rangle = \mathbf{0}$ in *any* stationary state, bound and unbound. But the counterexample $V(\mathbf{x}) \equiv 0$, $\psi(\mathbf{x}) = \exp(i\mathbf{k} \cdot \mathbf{x})$ shows that there is a mistake in the argument. What is it?

Exercise 4. For the ground state of the hydrogen atom, calculate the probability that the electron and the proton are farther apart than would be permitted by classical mechanics at the same total energy.

Exercise 5. *Bonus*–This exercise is beyond the scope of the final.

a) Consider a singly-ionized helium ion. How much more energy does it take to ionize its bound electron compared to hydrogen?

b) Still with He^+. What is the wavelength of the emitted photon during the electron transition from $n = 2 \to 1$?

c) Now, consider the usual helium-4. Which ground state has higher energy, parahelium (spin singlet) or orthohelium (spin triplet)? Why? ***Griffiths 5.14.*** How would this change if the two electrons are identical bosons?

d) Helium-3 is a fermion with spin-1/2 as compared to helium-4, which is a boson. Why?