Week 1 Worksheet Solutions
Math Review and de Broglie

Jacob Erlikhman

June 23, 2026

Exercise 1. Probability. Suppose you drop a match of length 1 cm onto a ruled sheet of paper whose lines are 1 cm apart. What is the probability that the match crosses a line on the paper?
Hints: First, figure out where the center of the match will land. Then, determine the angle of the match relative to the lines on the paper.

Let's follow the suggestion of the hint. The probability that the center of the match is within a distance d from one of the lines is $\frac{d}{0.5} = 2d = \sin(\theta)$ for some angle θ between the match and the normal to the lines. Given that the center is within a distance d from one of the lines, the probability that it hits one of the lines is then $2\theta/\pi$. Thus, the total probability of a hit is (we'll make sure it's positive after we evaluate the integral)

$$\int_0^{\pi/2} \frac{2\theta}{\pi} \sin(\theta) d\theta = -\frac{2}{\pi} \int_0^{\pi/2} \cos\theta d\theta = -\frac{2}{\pi}.$$

So the answer is $2/\pi$.

Exercise 2. de Broglie Formula. In this exercise, you will obtain the formula

$$\begin{aligned} p =& \hbar k \\ =& \frac{h}{\lambda}. \end{aligned}$$

a) The group velocity of a wave packet is defined as

$$\begin{aligned} v =& \frac{d\omega}{dk} \\ =& \frac{df}{d(1/\lambda)}. \end{aligned}$$

Differentiate the formula for the energy of a particle with frequency f, $E = hf$, to obtain a formula for $dE/d\lambda$ in terms of v and λ.

b) Using $E = \gamma m$ and $p = \gamma m v$, integrate your formula from (a) to get $p = h/\lambda$.

Hints: A trig substitution helps to evaluate the integral in (b), and $d(1/\lambda)$ can be evaluated using the chain rule. Recall that

$$\gamma \equiv \frac{1}{\sqrt{1-v^2}}$$

in natural units.

a) We differentiate $E = hf$ with respect to λ to get

$$\frac{dE}{d\lambda} = h\frac{df}{d\lambda}.$$

Since

$$d(1/\lambda) = -\frac{1}{\lambda^2}d\lambda,$$

we can write

$$\frac{dE}{d\lambda} = -\frac{h}{\lambda^2}v.$$

b) Note that v is a function of E in this expression, via the formula $E = \gamma m$! We want to integrate both sides, but first we need to determine the relationship between dE and dv. Taking the differential of $E = \gamma m$, we find

$$dE = m\gamma^3 v dv.$$

Thus, the differential equation we want to solve becomes

$$m\gamma^3 v\frac{dv}{d\lambda} = -\frac{h}{\lambda^2}v.$$

This separates as

$$m\gamma^3 dv = -\frac{h}{\lambda^2}d\lambda.$$

To do the integral on the LHS, write $v = \sin\theta$, so that $dv = \cos(\theta)d\theta$. Thus, we can evaluate the LHS as

$$m\int \sec^2\theta d\theta = m\tan\theta.$$

If $v = \sin\theta$, then

$$\tan\theta = \gamma v,$$

so the LHS becomes $m\gamma v = p$. On the other hand, the RHS becomes

$$-\int \frac{h}{\lambda^2}d\lambda = \frac{h}{\lambda},$$

as desired.

Exercise 3. de Broglie Wavelengths by Numbers. Calculate the de Broglie wavelength of the following particles.

a) A mass of 1 g moving with a velocity of 1 m/s.

b) A free electron with a kinetic energy of 200 eV.

c) A free electron with a kinetic energy of 50 GeV.

d) A free proton with a kinetic energy of 200 eV.

Hints: Use $hc = 1240$ eV nm, and note that in (c) you need to use the relativistic formula for momentum.

a) The de Broglie wavelength is $\lambda = h/p$. The momentum is 1 g m/s, so

$$\begin{aligned}\lambda =& \frac{hc}{pc} \\ =& \frac{1240 \text{ eV nm}}{3 \cdot 10^5 \text{ kg(m/s)}^2} \\ =& 4.1 \cdot 10^{-22} \text{ nm.}\end{aligned}$$

b) We can use $p = \sqrt{2mE}$ since the energy is not relativistic. Since $m = 511$ keV, we have

$$\frac{hc}{\sqrt{2 \cdot 511 \text{ keV} \cdot 200 \text{ eV}}} = \frac{1240 \text{ eV nm}}{\sqrt{2} \cdot 10^4 \text{ eV}} = 0.18 \text{ nm.}$$

c) For this one, we need to use the relativistic energy $E^2 = p^2 + m^2$, but we may as well approximate $E \sim p$, since $p \gg m$. So we have

$$\frac{hc}{E} = \frac{1240 \text{ eV nm}}{50 \cdot 10^9 \text{ eV}} = 24.8 \text{ nm.}$$

d) Again, this proton isn't relativistic, so, as in (b), we write

$$\frac{hc}{\sqrt{2 \cdot 1 \text{ GeV} \cdot 200 \text{ eV}}} = \frac{1240 \text{ eV nm}}{20 \cdot 10^4 \sqrt{10} \text{ eV}} = \frac{6.2}{\sqrt{10}} \cdot 10^{-3} \text{ nm.}$$