Week 2 Worksheet 2 Solutions

Jacob Erlikhman

July 1, 2026

Exercise 1. Consider the "even" infinite square well: $V(x) = 0$ for $x \in (-a, a)$.

a) Solve the Schrödinger equation for this potential, and obtain the complete basis of solutions.

b) Explain how you could have gotten the solutions from the usual infinite square well ($V(x) = 0$ for $x \in (0, a)$) by a change of coordinates. Check that this gives the same answer as (a).

a) We just need to impose the boundary conditions that $\psi(a) = \psi(-a) = 0$. Then we can observe that the solutions are

$$\sin\left(\frac{n\pi x}{a}\right)$$

with n any integer and

$$\cos\left(\frac{m\pi x}{2a}\right),$$

with m any odd integer. We can make these more symmetric by instead writing

$$\sin\left(\frac{m\pi x}{2a}\right) \quad \text{and} \quad \cos\left(\frac{m\pi x}{2a}\right),$$

where m is even for sine and odd for the cosine. These then have energies given by

$$\frac{m^2\pi^2\hbar^2}{8ma^2}.$$

b) We could have made the change of variables $x \to \frac{x+a}{2}$. Then the solutions to the usual infinite square well become

$$\sin\left(\frac{n\pi(x+a)}{2a}\right) = \sin\left(\frac{n\pi x}{2a} + \frac{n\pi}{2}\right).$$

It's then easy to see that when n is even we get the formula for the sine solutions and when n is odd we get the formula for the cosine solutions.

Exercise 2 (Introduction to Coherent States)**.** In this problem, we will study minimum uncertainty states, also called coherent states. Consider the ground state of the harmonic oscillator,

$$\psi_0(x) = Ae^{-m\omega x^2/2\hbar}.$$

a) Normalize $\psi_0(x)$.

b) Show that this state has $\langle x\rangle = \langle p\rangle = 0$.

c) Show that this state also has $\sigma_x\sigma_p = \hbar/2$, so it has **minimum uncertainty**. It is called a **coherent state**.

a) Set

$$\frac{m\omega}{2\hbar} = \alpha.$$

Then

$$\begin{aligned}\int |\psi_0|^2 dx =& |A|^2 \int e^{-2\alpha x^2} dx \\ =& |A|^2 \sqrt{\frac{\pi}{2\alpha}}.\end{aligned}$$

Thus,

$$A = \left(\frac{2\alpha}{\pi}\right)^{1/4}.$$

b) Since ψ_0 is even and both $x\psi_0$ and $p\psi_0$ are odd, we see that $\langle x\rangle = \langle p\rangle = 0$.

c) We need to compute $\langle x^2\rangle$ and $\langle p^2\rangle$. For the former, we have

$$\begin{aligned}\sqrt{\frac{2\alpha}{\pi}} \int x^2 e^{-2\alpha x^2} dx =& \sqrt{\frac{2\alpha}{\pi}} \frac{\sqrt{\pi}}{2(2\alpha)^{3/2}} \\ =& \frac{1}{4\alpha}.\end{aligned}$$

For $\langle p^2\rangle$, it's convenient to notice that $\langle p^2\rangle = \int |p\psi_0|^2 dx$. We compute

$$p\psi_0(x) = 2i\hbar\alpha x\psi_0(x).$$

Thus,

$$\begin{aligned}\langle p^2\rangle =& \sqrt{\frac{2\alpha}{\pi}} 4\hbar^2\alpha^2 \int x^2 e^{-2\alpha x^2} dx \\ =& \sqrt{\frac{2\alpha}{\pi}} 4\hbar^2\alpha^2 \frac{\sqrt{\pi}}{2(2\alpha)^{3/2}} \\ =& \hbar^2\alpha.\end{aligned}$$

Since $\sigma_x = \sqrt{\langle x\rangle^2}$ and $\sigma_p = \sqrt{\langle p\rangle^2}$, we immediately compute

$$\sigma_x\sigma_p = \hbar/2.$$