Week 3 Worksheet 1 Solutions

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Exercise 1. The hamiltonian for the harmonic oscillator is

$$H = \frac{1}{2m}[p^2 + (m\omega x)^2].$$

a) Show that

$$H = \hbar\omega(a^\dagger a + 1/2),$$

where

$$\begin{aligned} a^\dagger &= \alpha x + i\beta p \\ a &= \alpha x - i\beta p. \end{aligned}$$

That is, determine α and β.

b) Show that $[a, a^\dagger] = aa^\dagger - a^\dagger a = 1$.

c) Show that $[a^\dagger a, a] = -a$.

d) If $N = a^\dagger a$ and $N\psi = \lambda\psi$ for some scalar λ and some normalizable ψ, show that $Na\psi = (\lambda - 1)a\psi$.

e) Show that if ψ is normalized, then

$$\int |a\psi|^2 dx = \int \left(\psi^*(\alpha x)^2\psi + \psi^*(\beta p)^2\psi\right) dx - 1/2.$$

Hint: Use integration by parts, noting that $(|\psi|^2)' = (\psi^*)'\psi + \psi^*\psi'$.

f) Show that there is a normalized solution ψ_0 to the TISE such that $a^\dagger a\psi_0 = 0$.
Hints: Use part (e) to show that $\int |a\psi|^2 dx = \int \psi^* N\psi dx$. Together with part (d), what does this tell you about repeated applications of a to ψ?

g) What is the energy of ψ_0?

h) Show that $N = a^\dagger a$ is the **number operator**: If $\psi_n = (a^\dagger)^n\psi_0$ is the n^{th} stationary state, then

$$N\psi_n = n\psi_n.$$

i) Show that if $N\psi_0 = 0$, then a complete set of normalized solutions to the equation $N\psi = \lambda\psi$ is given by $(a^\dagger)^n\psi_0$, so that $(a^\dagger)^n\psi_0$ are also a complete set of solutions to the TISE.

a) We compute

$$a^\dagger a = \alpha^2 x^2 + \beta^2 p^2 + \hbar\alpha\beta.$$

From this we can read off that

$$\begin{aligned}|\alpha| =& \sqrt{\frac{m\omega}{2\hbar}}\\ |\beta| =& \frac{1}{\sqrt{2m\hbar\omega}}.\end{aligned}$$

To determine the signs, we can solve the equation

$$\hbar\alpha\beta + \frac{1}{2} = 0.$$

This has solution

$$\alpha\beta = -\frac{1}{2\hbar}.$$

If we want to keep α positive, then we set

$$\beta = -\frac{1}{\sqrt{2m\hbar\omega}}.$$

b) We compute

$$[a, a^\dagger] = -2\hbar\alpha\beta = 1.$$

c) We compute

$$\begin{aligned}[N, a] =& a^\dagger a^2 - a a^\dagger a\\ =& [a^\dagger, a]a\\ =& -a.\end{aligned}$$

d) Using (c), we have

$$Na\psi = (-a + aN)\psi = (\lambda - 1)a\psi.$$

e) We can compute that

$$a\psi = \alpha x\psi - i\beta p\psi = \alpha x\psi - \hbar\beta\psi'.$$

In particular, notice that all the coefficients in front of ψ or ψ' are real. Thus, $(a\psi)^* = a\psi^*$, and we can compute

$$|a\psi|^2 = \psi^*(\alpha x)^2\psi + (\hbar\beta)^2|\psi'|^2 + \frac{x}{2}\left((\psi^*)'\psi + \psi^*\psi'\right).$$

Following the hint, we can write the last term as

$$\frac{x}{2}\left(|\psi|^2\right)',$$

which can be integrated by parts to get the $-1/2$ to appear. The first term is already in the right form, while the second term can be written

$$(\hbar\beta)^2|\psi'|^2 = \beta^2[i\hbar(\psi^*)'][-i\hbar\psi'].$$

This can again be integrated by parts to pick up a minus sign and get

$$\beta^2\psi^* p^2\psi.$$

f) Part (d) means that repeated applications of a lower the eigenvalue of ψ under N. To show that

$$\int |a\psi|^2 dx = \int \psi^* N\psi dx,$$

we just need to observe that

$$a^\dagger a\psi = (\alpha x)^2\psi + (\beta p)^2\psi - \psi/2,$$

which is exactly the form of (c). Now, if $\int |a^n\psi|^2 dx = \int (a^{n-1}\psi)^* N a^{n-1}\psi dx = \int (\lambda - n - 1)|a^{n-1}\psi|^2 dx$, that means that there will be some large n such that $\lambda - n - 1 \leq 0$. Since $\int |a^n\psi|^2 dx \geq 0$, we find that $\lambda - n - 1 = 0$ for some large n. Thus, there is a state with $\lambda' = \lambda - n - 1 = 0$.

g) It's $\hbar\omega/2$, since $N\psi_0 = 0$.

h) We compute

$$N(a^\dagger)^n\psi_0 = a^\dagger a a^\dagger (a^\dagger)^{n-1}\psi_0 = (a^\dagger)^n\psi_0 + (a^\dagger)^2 a (a^\dagger)^{n-1}\psi_0 = 2(a^\dagger)^n\psi_0 + (a^\dagger)^3 a (a^\dagger)^{n-2}\psi_0.$$

At this point, we can see the pattern appearing: Eventually, we will move the a all the way to the right, which will kill ψ_0, and this will happen precisely when the first term looks like

$$n(a^\dagger)^n\psi_0.$$

i) We already know that the "lowest" solution is ψ_0, and we know that any $\psi_n = (a^\dagger)^n\psi_0$ is a solution. There can't be any others, since if λ is not a nonnegative integer, then part (d) shows that there is no solution to the equation $N\psi = \lambda\psi$.