Week 3 Worksheet 1

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Exercise 1. The hamiltonian for the harmonic oscillator is

$$H = \frac{1}{2m}[p^2 + (m\omega x)^2].$$

a) Show that

$$H = \hbar\omega(a^\dagger a + 1/2),$$

where

$$\begin{aligned} a^\dagger =& \alpha x + i\beta p \\ a =& \alpha x - i\beta p. \end{aligned}$$

That is, determine α and β.

b) Show that $[a, a^\dagger] = aa^\dagger - a^\dagger a = 1$.

c) Show that $[a^\dagger a, a] = -a$.

d) If $N = a^\dagger a$ and $N\psi = \lambda\psi$ for some scalar λ and some normalizable ψ, show that $Na\psi = (\lambda - 1)a\psi$.

e) Show that if ψ is normalized, then

$$\int |a\psi|^2 dx = \int \left(\psi^*(\alpha x)^2\psi + \psi^*(\beta p)^2\psi\right) dx - 1/2.$$

Hint: Use integration by parts, noting that $(|\psi|^2)' = (\psi^*)'\psi + \psi^*\psi'$.

f) Show that there is a normalized solution ψ_0 to the TISE such that $a^\dagger a\psi_0 = 0$.
Hints: Use part (e) to show that $\int |a\psi|^2 dx = \int \psi^* N\psi dx$. Together with part (d), what does this tell you about repeated applications of a to ψ?

g) What is the energy of ψ_0?

h) Show that $N = a^\dagger a$ is the **number operator**: If $\psi_n = (a^\dagger)^n\psi_0$ is the n^{th} stationary state, then

$$N\psi_n = n\psi_n.$$

i) Show that if $N\psi_0 = 0$, then a complete set of normalized solutions to the equation $N\psi = \lambda\psi$ is given by $(a^\dagger)^n\psi_0$, so that $(a^\dagger)^n\psi_0$ are also a complete set of solutions to the TISE.