Week 4 Worksheet 1 Solutions

Jacob Erlikhman

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Exercise 1. Suppose V is a (complex) vector space. Let V^* be the space of all *linear* maps $V \to \mathbb{C}$. A map $\varphi : V \to \mathbb{C}$ is called **linear** if

$$\varphi(a\psi_1 + b\psi_2) = a\varphi(\psi_1) + b\varphi(\psi_2)$$

for any ψ_i in V and any complex numbers a and b.

a) If V has dimension $n < \infty$, show that V^* is also a vector space and has dimension n.
Hint: Use a basis for V to construct a basis for V^*.

b) Recall that if V is a vector space, a (complex-valued) **inner product** on V assigns to any ordered pair of two vectors ψ_1, ψ_2 in V a complex number $\langle\psi_1|\psi_2\rangle = \langle\psi_2|\psi_1\rangle^*$ such that if $\psi \neq 0$, then $\langle\psi|\psi\rangle > 0$. Show that in the situation of (a), an inner product on V is the same thing as a *bi*linear map, i.e. a map which is linear in both inputs, $V^* \times V \to \mathbb{C}$ such that its value is 0 if and only if one of the inputs is 0 (we say it's **nondegenerate**).

c) Let $L^2(\mathbb{R})$ be the vector space of complex-valued, square-normalizable functions on the real line, $\psi : \mathbb{R} \to \mathbb{C}$. A function $\psi : \mathbb{R} \to \mathbb{C}$ is **square-normalizable** if

$$\int_{-\infty}^{\infty} \psi^*(x)\psi(x)dx < \infty.$$

Explain how $L^2(\mathbb{R})$ is a vector space.

d) For the vector space $V = L^2(\mathbb{R})$ of (c), show that there are functions in V^* which are not in V. This means V^* is *strictly larger* than V.
Hint: Consider a function of the form

$$\psi(x) \to \int \varphi(x)\psi(x)dx.$$

e) What would go wrong if you tried to apply your argument for (a) to the vector space $L^2(\mathbb{R})$?

f) Check that the assignment

$$\langle\psi_1|\psi_2\rangle = \int {\psi_1}^*\psi_2 dx$$

is an inner product on $L^2(\mathbb{R})$.

g) Is there a bilinear map $L^2(\mathbb{R})^* \times L^2(\mathbb{R}) \to \mathbb{C}$ which recovers the inner product of (f)?

a) Let $\{\psi_i\}$ be a basis of V. Define elements φ_i in V^* by

$$\varphi_i(\psi_j) = \delta_{ij}.$$

Given a general function $\varphi \in V^*$, let

$$\varphi(\psi_i) = a_i.$$

Then $\varphi = \sum a_i \varphi_i$, so that $\{\varphi_i\}$ is a basis for V^*. This implies that V^* has dimension n as a vector space (because the basis is n elements). The basis elements are linearly independent because they have different values on at least on of the ψ_i's. Checking the vector space axioms for V^*: The 0-function is in V^*, and $0+\varphi = \varphi$. Scalar multiplication and vector addition work as intended because elements of V^* are linear functions.

b) If we have an inner product $\langle - | - \rangle$, we see that if we fix a vector $\psi \in V$, then $\langle \psi | - \rangle$ defines an element of V^*. So we can define a bilinear function

$$(\varphi, \psi) = (\sum a_i \varphi_i, \psi) \to \langle \sum a_i \psi_i | \psi \rangle .$$

To go the other way, given a bilinear function $B : V^* \times V \to \mathbb{C}$, we can define an inner product by setting

$$\langle \psi, \psi' \rangle = B(\sum a_i \varphi_i, \psi'),$$

where $\psi = \sum a_i \psi_i$.

c) We need to check that sums of functions and scalar multiples of functions are in this space, since all the other vector space axioms are easily satisfied. We check

$$|\psi_1 + \psi_2|^2 \leq |\psi_1|^2 + |\psi_2|^2$$

by the triangle identity. This implies that if $\psi_1, \psi_2 \in L^2(\mathbb{R})$, then so is $\psi_1 + \psi_2$. Scalar multiples can't send an integral to infinity, so we're done.

d) The function

$$\psi(x) \to \int \delta(x)\psi(x)dx$$

is not in $L^2(\mathbb{R})$, since $\delta(x)$ is not square-normalizable.

e) We might never get a complete basis for V^* even though we have one for V. In other words, given a basis $\{\psi_i\}$, the dual vectors $\{\varphi_i\}$ won't form a complete basis for V^*. This is because both V and V^* are infinite-dimensional, so it's possible that V^* is bigger than V while staying the same size. This is what the previous part shows.

f) Bilinearity is obvious by the distributive law. To see nondegeneracy, just notice that $\psi^*\psi = |\psi|^2 \geq 0$, and it's equal to 0 if and only if $\psi \equiv 0$.

g) No. The reason we shouldn't expect there to be one is because the argument of (b) used the correspondence of (a) in order to construct the bilinear function $V^* \times V \to \mathbb{C}$. Since by the parts (d-e) we don't have access to the correspondence of part (a), we can't construct a bilinear function as desired.