Week 4 Worksheet 1

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Exercise 1. Suppose V is a (complex) vector space. Let V^* be the space of all *linear* maps $V \to \mathbb{C}$. A map $\varphi : V \to \mathbb{C}$ is called **linear** if

$$\varphi(a\psi_1 + b\psi_2) = a\varphi(\psi_1) + b\varphi(\psi_2)$$

for any ψ_i in V and any complex numbers a and b.

a) If V has dimension $n < \infty$, show that V^* is also a vector space and has dimension n.
Hint: Use a basis for V to construct a basis for V^*.

b) Recall that if V is a vector space, a (complex-valued) **inner product** on V assigns to any ordered pair of two vectors ψ_1, ψ_2 in V a complex number $\langle\psi_1|\psi_2\rangle = \langle\psi_2|\psi_1\rangle^*$ such that if $\psi \neq 0$, then $\langle\psi|\psi\rangle > 0$. Show that in the situation of (a), an inner product on V is the same thing as a *bi*linear map, i.e. a map which is linear in both inputs, $V^* \times V \to \mathbb{C}$ such that its value is 0 if and only if one of the inputs is 0 (we say it's **nondegenerate**).

c) Let $L^2(\mathbb{R})$ be the vector space of complex-valued, square-normalizable functions on the real line, $\psi : \mathbb{R} \to \mathbb{C}$. A function $\psi : \mathbb{R} \to \mathbb{C}$ is **square-normalizable** if

$$\int_{-\infty}^{\infty} \psi^*(x)\psi(x)dx < \infty.$$

Explain how $L^2(\mathbb{R})$ is a vector space.

d) For the vector space $V = L^2(\mathbb{R})$ of (c), show that there are functions in V^* which are not in V. This means V^* is *strictly larger* than V.
Hint: Consider a function of the form

$$\psi(x) \to \int \varphi(x)\psi(x)dx.$$

e) What would go wrong if you tried to apply your argument for (a) to the vector space $L^2(\mathbb{R})$?

f) Check that the assignment

$$\langle\psi_1|\psi_2\rangle = \int {\psi_1}^*\psi_2 dx$$

is an inner product on $L^2(\mathbb{R})$.

g) Is there a bilinear map $L^2(\mathbb{R})^* \times L^2(\mathbb{R}) \to \mathbb{C}$ which recovers the inner product of (f)?