Week 5 Worksheet 2

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Exercise 1. Denote the eigenspinors of $S_z = \frac{\hbar}{2}\sigma_z$ for a spin-1/2 particle by $|+\rangle$ and $|-\rangle$, with eigenvalues $\hbar/2$ and $-\hbar/2$, respectively.

a) Write down $S_y = \frac{\hbar}{2}\sigma_y$ and $S_x = \frac{\hbar}{2}\sigma_x$ as

$$a\,|+\rangle\,\langle+| + b\,|+\rangle\,\langle-| + c\,|-\rangle\,\langle+| + d\,|-\rangle\,\langle-|\,;$$

that is, determine a, b, c, d.

b) Explain how you could extract a, b, c, d for S_i by taking $\langle\pm|S_i|\pm\rangle$ or $\langle\pm|S_i|\mp\rangle$.

Exercise 2. ***Adapted from Griffiths 4.33*** Suppose we instead wanted to compute spin for a spin-1/2 particle along an arbitrary direction $\hat{n}$ (the hats in this problem denote "unit vector," not "operator").

a) Convince yourself that we can place the z-axis along $\hat{n}$ by rotating first around $\hat{y}$ by an angle β and then around $\hat{z}$ (the old z-axis, not the new one!) by an angle α.
Hint: Write $\hat{n}$ in terms of its polar and azimuthal angles. How are these related to α and β?

b) Use this to compute the eigenspinor of S_n with eigenvalue $\hbar/2$, $|\hat{n};+\rangle$: It is given by

$$|\hat{n};+\rangle = e^{-\frac{i}{2}\alpha\sigma_z}e^{-\frac{i}{2}\beta\sigma_y}\,|+\rangle\,,$$

where $|+\rangle$ is the eigenvector of S_z with eigenvalue $\hbar/2$.
Hint: You can use the formula which you will prove on the review:

$$e^{-i\theta\hat{n}\cdot\boldsymbol{\sigma}} = \mathbb{1}\cos\theta - i\,(\hat{n}\cdot\boldsymbol{\sigma})\sin\theta.$$

c) Check that the spinor $|\hat{n};+\rangle$ is an eigenspinor of $\hat{n}\cdot\boldsymbol{\sigma}$, which shows it is indeed an eigenspinor of $S_n = \hat{n}\cdot\mathbf{S}$.
Hints: Use the following three facts you will prove on the review: If H is hermitian, then e^{iH} is unitary. If U_1, U_2 are two unitary operators, then U_1U_2 is also unitary. If $U = e^{-\frac{i}{2}\alpha\sigma_z}e^{-\frac{i}{2}\beta\sigma_y}$, then

$$U^\dagger\boldsymbol{\sigma}U = R\boldsymbol{\sigma}\,,$$

where R is the (3×3)-matrix which rotates the z-axis to $\hat{n}$. Also note that if $\mathbf{a}, \mathbf{b}$ are 3-vectors and R is an invertible (3×3)-matrix, then $\mathbf{a}\cdot R\mathbf{b} = R^{-1}\mathbf{a}\cdot\mathbf{b}$.

d) Repeat (b) and (c) for $|\hat{n};-\rangle$.