

Week 7 Worksheet 2

Identical Particles

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August 5, 2026

Exercise 1. Griffiths 5.5a. Write down the hamiltonian for two noninteracting identical particles in the infinite square well. Write down the ground states for the three cases: distinguishable, fermions, bosons. Recall that the one-particle wavefunctions are

$$\psi_n(x) = \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi}{a}x\right),$$

with energies $E_n = n^2\pi^2\hbar^2/2ma^2$.

Exercise 2. Griffiths 5.9. In Exercise 1, we ignored spin (or at least supposed that the particles are in the same spin state).

- a) Do it now for particles of spin 1/2. Construct the four lowest-energy configurations, and specify their energies and degeneracies.

Hint: Recall that the *total* state vector for a boson (resp. fermion) must be symmetric (resp. anti-symmetric). If a boson or fermion state vector is a product of two vectors (e.g. a spatial state vector and a spin state vector), can these components be symmetric, anti-symmetric, or both?

- b) Do the same for spin 1.

Hint: You can do this without having to use any Clebsch-Gordan coefficients!

Exercise 3. (Bonus/Challenge) Symmetries of Many-Particle States.

- a) Consider a system of two identical particles. Define the operator P_{12} via

$$P_{12} |a\rangle |b\rangle = |b\rangle |a\rangle.$$

Show that $P_{12}^2 = 1$, the identity operator, and that the eigenvalues of P_{12} are ± 1 . Thus, show that its eigenvectors are either totally symmetric or antisymmetric. We call P_{12} a **permutation operator**. In this case, there are only two such operators: P_{12} and $P_{12}^2 = 1$.

- b) Generalize part (a) to systems of three identical particles. You should find that you have *six* permutation operators (note that the identity is a permutation operator). Assuming the hamiltonian is invariant under each of these operators, is there a complete set of common eigenvectors?

- c) **Griffiths 5.8.** In the situation of (b), suppose that the particles have access to three distinct one-particle states, $|a\rangle$, $|b\rangle$, and $|c\rangle$. For example, $|abc\rangle$ is an allowed state, as is $|aaa\rangle$. How many states can be constructed if they are (i) bosons or (ii) fermions?
- d) Suppose we have a single-particle fermion state $|\alpha\rangle$ and a single-particle bosonic state $|\beta\rangle$. Just like for the harmonic oscillator, we can define **creation operators** $C_\alpha^\dagger$ and $a_\beta^\dagger$, such that given any state $|\psi\rangle$,

$$\begin{aligned} C_\alpha^\dagger |\psi\rangle &= |\alpha\psi\rangle \\ a_\beta^\dagger |\psi\rangle &= |\beta\psi\rangle. \end{aligned}$$

The operators $C_\alpha^\dagger$ and $a_\beta^\dagger$ have the following properties. You don't need to prove them.

$$\begin{aligned} C_\alpha |\alpha\psi\rangle &= |\psi\rangle \\ a_\beta |\beta\psi\rangle &= |\psi\rangle \\ C_\alpha |0\rangle &= a_\beta |0\rangle = 0 \\ C_\alpha^\dagger C_\alpha^\dagger &= 0 \\ \{C_\alpha, C_{\alpha'}^\dagger\} &\equiv C_\alpha C_{\alpha'}^\dagger + C_{\alpha'}^\dagger C_\alpha = \delta_{\alpha\alpha'} 1 \\ \{C_\alpha^\dagger, C_{\alpha'}^\dagger\} &= 0 \\ [a_\beta, a_{\beta'}^\dagger] &= \delta_{\beta\beta'} 1 \\ [a_\beta^\dagger, a_{\beta'}^\dagger] &= 0, \end{aligned}$$

where $|0\rangle$ denotes a state with no particles at all. To what extent is a bound pair of fermions equivalent to a boson?

Hint: Use the symmetries of many-particle states and the (anti-)commutation relations of the creation/annihilation operators constructed in parts (a)-(d). What algebra must the creation/annihilation operators for the bound pair satisfy?

- e) **Challenge.** Prove the properties given in (d).

Hints: It may be useful to use the notation $\sim \alpha$ for the α “orbital” being *unoccupied*. To show the first relation for C_α , try to first show that $C_\alpha |\alpha\rangle = |0\rangle$. For the anti-commutator relations, consider separately the cases $\alpha \neq \alpha'$ and whether the α or α' orbitals are occupied.