Week 8 Worksheet 1

Jacob Erlikhman

August 10, 2026

Exercise 1. *Hidden Symmetries.* In classical mechanics, $\frac{1}{r}$ potentials have an additional conserved quantity that is rarely covered in introductory courses. This quantity is called the **Runge-Lenz vector**, and it is given by

$$\mathbf{F} = \frac{1}{m}\mathbf{p} \times \mathbf{L} - \frac{\gamma}{r}\mathbf{r},$$

where γ is the constant associated to the potential $V(r) = -\gamma/r$, e.g. $\gamma = e^2/4\pi\varepsilon_0$ or $\gamma = MG$.

a) If we replace all the classical dynamical variables in the above expression by quantum operators, explain why the result is ambiguous.
Hints: When we upgrade $\mathbf{L} = \mathbf{r} \times \mathbf{p}$ to a quantum operator, note that $\mathbf{L} = -\mathbf{p} \times \mathbf{r}$ as operators. Why? Does $\mathbf{p} \times \mathbf{L} = -\mathbf{L} \times \mathbf{p}$ as operators?

b) It turns out[1] that the correct quantum mechanical version of $\mathbf{F}$ is

$$\mathbf{F} = \frac{1}{2m}\left(\mathbf{p} \times \mathbf{L} - \mathbf{L} \times \mathbf{p}\right) - \frac{\gamma}{r}\mathbf{r}.$$

Show that for the hydrogen atom hamiltonian $H = p^2/2m - \gamma/r$, $[H, \mathbf{F}] = \mathbf{0}$, so that $\mathbf{F}$ is a symmetry.
Hints: Compute separately the commutators $[1/r, (\mathbf{p} \times \mathbf{L})_k - (\mathbf{L} \times \mathbf{p})_k]$ and $[p^2, r_k/r]$; then, show that they exactly cancel each other. Since $1/r$ is rotationally symmetric, what does that tell you about the commutator $[1/r, L_j]$? Recall "BAC-CAB:"

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b}(\mathbf{a} \cdot \mathbf{c}) - \mathbf{c}(\mathbf{a} \cdot \mathbf{b}).$$

What is the analog of this formula in quantum mechanics, when $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are operators (and don't necessarily commute)?

[1] The way you would prove this is by matching Poisson bracket relations with $\mathbf{F}$ in classical mechanics to corresponding ones in quantum mechanics (upgrading the Poisson brackets to commutators). This would then allow you to determine the right combination of $\mathbf{p} \times \mathbf{L}$ and $\mathbf{L} \times \mathbf{p}$ to take.