

Midterm 1 Review Solutions

Exercise 1. We can represent a counter-clockwise rotation by an angle θ about an axis \hat{r} by the unitary operator

$$U(\theta) = e^{-i\theta\hat{r}\cdot\mathbf{S}/\hbar},$$

where \mathbf{S} is the angular momentum operator. For particles of spin 1/2, $\mathbf{S} = \hbar\boldsymbol{\sigma}/2$.

a) Show that $(\hat{r} \cdot \boldsymbol{\sigma})^2 = \mathbb{1}$, the identity operator.

Hint: Using $[\sigma_i, \sigma_j] = 2i\varepsilon_{ijk}\sigma_k$ and $\{\sigma_i, \sigma_j\} = 2\delta_{ij}\mathbb{1}$, show first that

$$\sigma_i\sigma_j = \mathbb{1}\delta_{ij} + i\varepsilon_{ijk}\sigma_k.$$

b) Show that

$$U(\theta) = \mathbb{1} \cos(\theta/2) - i\hat{r} \cdot \boldsymbol{\sigma} \sin(\theta/2).$$

c) Determine the spin operator σ_θ which points in the direction described by (θ, φ) with $\varphi = 0$.

Hint: Do this by rotating σ_z by an angle θ about the y -axis.

d) Redo problem 4.59 from Griffiths: If two electrons are in the spin singlet state, $S_z^{(1)}$ is the component of spin angular momentum of particle 1 along the z -axis, and $S_\theta^{(2)}$ is the spin angular momentum of particle 2 along the $\hat{r} = (\theta, 0)$ axis, show that

$$\left\langle S_z^{(1)} S_\theta^{(2)} \right\rangle = -\frac{\hbar^2}{4} \cos \theta.$$

a) Add the commutator to the anti-commutator to get the relation given in the hint. Now,

$$(\hat{r} \cdot \boldsymbol{\sigma})^2 = r_i\sigma_i r_j\sigma_j = r_i r_j (\mathbb{1}\delta_{ij} + i\varepsilon_{ijk}\sigma_k) = \mathbb{1},$$

since $\hat{r}^2 = \sum r_i^2 = 1$ and ε_{ijk} is antisymmetric under interchange of indices, while $r_i r_j$ is symmetric.

b) Write

$$U(\theta) = e^{-i\theta\hat{r}\cdot\mathbf{S}/\hbar},$$

and expand the exponential as a power series:

$$\begin{aligned} U(\theta) &= \sum_{n=0}^{\infty} \frac{(-i\theta/2)^n (\hat{r} \cdot \boldsymbol{\sigma})^n}{n!} \\ &= \sum_{k=0}^{\infty} \frac{(-i\theta/2)^{2k}}{(2k)!} \mathbb{1} + \sum_{k=0}^{\infty} \frac{(-i\theta/2)^{2k+1}}{(2k+1)!} \hat{r} \cdot \boldsymbol{\sigma} \\ &= \mathbb{1} \cos(\theta/2) - i \hat{r} \cdot \boldsymbol{\sigma} \sin(\theta/2), \end{aligned}$$

where we use the power series for sine and cosine in the final line (and note that $(-i)^{2k} = (-1)^k$ and $(-i)^{2k+1} = -i(-1)^k$).

c) We have determined a nice form for the operator $U(\theta)$. Setting $\hat{r} = \hat{y}$, we have

$$U_y(\theta) = \mathbb{1} \cos(\theta/2) - i \sigma_y \sin(\theta/2).$$

A coordinate transformation which corresponds to $U_y(\theta)$ will transform operators written in the original basis by

$$A \mapsto U_y(\theta)^\dagger A U_y(\theta).$$

Thus, we have

$$\begin{aligned} \sigma_\theta &= U_y(\theta)^\dagger \sigma_z U_y(\theta) \\ &= \sigma_z \cos^2(\theta/2) - i [\sigma_z, \sigma_y] \sin(\theta/2) \cos(\theta/2) + \sigma_y \sigma_z \sigma_y \sin^2(\theta/2) \\ &= \sigma_z [\cos^2(\theta/2) - \sin^2(\theta/2)] - 2\sigma_x \sin(\theta/2) \cos(\theta/2) \\ &= \sigma_z \cos \theta - \sigma_x \sin \theta \\ &= \begin{bmatrix} \cos \theta & -\sin \theta \\ -\sin \theta & -\cos \theta \end{bmatrix}. \end{aligned}$$

d) Our spin state is

$$\frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle).$$

It's easiest to figure out what to do if we write this as vectors:

$$\frac{1}{\sqrt{2}} \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right).$$

Thus, we get

$$\begin{aligned} \langle S_z^{(1)} S_\theta^{(2)} \rangle &= \frac{\hbar^2}{8} \left(\begin{bmatrix} 1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \end{bmatrix} \right) \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} -\sin \theta \\ -\cos \theta \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \otimes \begin{bmatrix} \cos \theta \\ -\sin \theta \end{bmatrix} \right) \\ &= \frac{\hbar^2}{8} (-\cos \theta - \cos \theta) \\ &= -\frac{\hbar^2}{4} \cos \theta. \end{aligned}$$

Exercise 2. Griffiths 5.9. Consider two non-interacting particles in an infinite square well of width a such that the single particle wavefunction is

$$\psi_n(x) = \sqrt{\frac{2}{a}} \sin(n\pi x/a)$$

with energy $E_n = n^2 K$. Construct the ground state and first excited state of the two-particle system if the particles are a) spin-1/2 and b) spin-1. Determine the energy and degeneracies of these states.

The solution to this problem is in the Week 3 Worksheet Solutions.

Exercise 3. Helium.

- Consider a singly-ionized helium ion. How much more energy does it take to ionize its bound electron compared to hydrogen?
- Still with He^+ . What is the wavelength of the emitted photon during the electron transition from $n = 2 \rightarrow 1$?
- Now, consider the usual helium-4. Which ground state has higher energy, parahelium (spin singlet) or orthohelium (spin triplet)? Why? **Griffiths 5.14**. How would this change if the two electrons are identical bosons?
- Griffiths 5.22**. Helium-3 is a fermion with spin-1/2 (as compared to helium-4, which is a boson. Why?). At low temperatures, helium-3 can be treated as a Fermi gas. If its mass density is 82 kg/m^3 , determine its Fermi temperature.
- The only difference between singly-ionized helium and hydrogen is the number of protons. The energy levels for hydrogen-like atoms come from a potential in the Schrödinger equation which is proportional to Ze^2 . Now, we have to remember that in hydrogen the energy levels go as Z^2 . Explicitly,

$$\begin{aligned} E_n &= -\frac{1}{2} \frac{\alpha^2 Z^2 mc^2}{n^2} \\ &= -E_0 \frac{Z^2}{n^2}, \end{aligned}$$

where $E_0 = 13.6 \text{ eV}$. Thus, helium-3 is 4 times harder to ionize than hydrogen.

- Recall that $E = pc$ for light; hence,

$$E = \frac{hc}{\lambda}.$$

Since $E_2 - E_1 = 3E_0$, we have

$$\lambda = \frac{hc}{3E_0}.$$

Since $hc = 1240 \text{ eV} \cdot \text{nm}$, we have

$$\lambda \approx 30 \text{ nm}.$$

- c) Since the triplet is symmetric, the spatial wavefunction which is associated to orthohelium must be antisymmetric. Since the lowest energy state is symmetric (both electrons in $n = 1$), it follows that the triplet has higher energy than the singlet. If they were both bosons, then we'd have the opposite.
- d) Helium-3 is a fermion because it has an odd number of fermions. The Fermi energy is given by (make sure you can derive this!)

$$E_F = \frac{\pi^2 \hbar^2}{2m} \left(\frac{3n}{\pi} \right)^{2/3},$$

where $n = N/V$ is the number density. Now, $mn = \rho$ and $E_F = kT_F$, where k is Boltzmann's constant, so

$$T_F = \frac{\hbar^2}{2} \left(\frac{3\rho\pi^2}{m^{5/2}} \right)^{2/3}.$$

We can now plug in numbers:

$$T_F \approx 4 \text{ K}.$$

Exercise 4. Consider a transformation on a physical system represented by a unitary operator U .

- a) How do kets transform under U ? What about operators?
- b) If the hamiltonian H commutes with U , what does that imply about H being invariant under the transformation U ? What does this imply about a non-degenerate eigenstate of H ?
- c) Derive parity selection rules for hydrogen with respect to momentum and angular momentum matrix elements. I.e. determine when

$$\langle n'l'm' | \mathbf{p} | nlm \rangle = 0$$

and

$$\langle n'l'm' | \mathbf{L} | nlm \rangle = 0.$$

- a) Kets just transform as

$$|\psi\rangle \mapsto U |\psi\rangle.$$

This question was intentionally ambiguous! If we consider passive transformations, then operators transform as

$$A \mapsto U^\dagger A U.$$

However, if we only consider active transformations and want to force $\langle A \rangle$ to be unchanged after action by U , then we want to send

$$A \mapsto U A U^\dagger.$$

Indeed, since

$$\langle A \rangle = \langle \psi | A | \psi \rangle,$$

we have

$$\langle A \rangle \mapsto \langle \psi | U^\dagger A' U | \psi \rangle,$$

so we need

$$A' = U A U^\dagger.$$

b) If the hamiltonian commutes with the symmetry U , then that means

$$UH = HU \implies UHU^\dagger = H.$$

Thus, if we have a nondegenerate eigenket of H with

$$H |\psi\rangle = \lambda |\psi\rangle,$$

then

$$HU |\psi\rangle = UH |\psi\rangle = U\lambda |\psi\rangle = \lambda U |\psi\rangle,$$

so that $U |\psi\rangle$ is also an eigenket of H .

c) We know that under parity $\hat{x} \mapsto -\hat{x}$, where \hat{x} denotes the position operator. Indeed, since an operator is determined by its action on a basis, we can take the basis $\{|x\rangle\}$ and consider the action of $\pi \hat{x} \pi$ on it.

$$\pi \hat{x} \pi |x\rangle = \pi \hat{x} |-x\rangle = -x |x\rangle,$$

so we must have $\pi \hat{x} \pi = -\hat{x}$. Similarly, momentum is odd under parity (dropping all the hats now):

$$[x, \pi p \pi] = x \pi p \pi - \pi p \pi x = -\pi x p \pi + \pi p x \pi = -\pi [x, p] \pi = -i\hbar,$$

so we must have $\pi p \pi = -p$. Now, we can write

$$\langle n'l'm' | \mathbf{p} | nlm \rangle = \langle n'l'm' | \pi \pi \mathbf{p} \pi \pi | nlm \rangle = -(-1)^{l+l'} \langle n'l'm' | \mathbf{p} | nlm \rangle.$$

Thus, this is exactly 0 for even values of $l + l'$. Similarly, we have

$$\langle n'l'm' | \mathbf{L} | nlm \rangle = (-1)^{l+l'} \langle n'l'm' | \mathbf{L} | nlm \rangle,$$

which is 0 for odd values of $l + l'$, since $\mathbf{L} = \mathbf{x} \times \mathbf{p}$ is even under parity (since both \mathbf{x} and \mathbf{p} are odd).

Exercise 5. Dilations. Do Exercise 2 on the Week 5 Worksheet: Another symmetry is called **dilation** symmetry. Dilations are given by the transformation $\mathbf{x} \rightarrow \mathbf{x}' = e^c \mathbf{x}$, where $c \in \mathbb{R}$. Call its generator D , so that e^{-icD} is the corresponding unitary operator.

Remark. In conformal field theory, the convention is to absorb the factor of i into D , so that e^{-cD} is the dilation operator.

a) Show that the *infinitesimal* transformation

$$e^{i\mathbf{a}\cdot\mathbf{p}} e^{icD} e^{-i\mathbf{a}\cdot\mathbf{p}} e^{-icD}$$

is given by $\mathbb{1} + c\mathbf{a} \cdot [D, \mathbf{p}]$.

Hints: You can reduce to the situation where all the vectors are 1-dimensional (why?). There's a slick way to do this (use the Baker-Campbell-Hausdorff formula), but the brute force method does work.

b) Calculate $[D, \mathbf{p}]$.

Hint: What coordinate transformation does the above correspond to? In other words, if you write it in the form $\mathbf{x} \rightarrow \mathbf{x}'$, what is \mathbf{x}' ?

a) You could write out all of these exponentials out to second order in a and c (note that it's easier to work in 1 dimension for this whole problem). Another (slicker) way to get the same answer is to use the Baker-Campbell-Hausdorff formula, which says that given any two operators X and Y , we have

$$e^X e^Y = e^Z,$$

where

$$Z = X + Y + \frac{1}{2}[X, Y] + \dots.$$

The ellipsis above denotes third and higher order terms in X and Y , which we can ignore. Thus, use BCH on

$$e^{-i\mathbf{a}\cdot\mathbf{p}} e^{-icD} = \exp\left(-i\mathbf{a}\cdot\mathbf{p} - icD + \frac{ac}{2}[D, \mathbf{p}] + \dots\right).$$

Then, use it on

$$e^{i\mathbf{a}\cdot\mathbf{p}} e^{icD} = \exp\left(i\mathbf{a}\cdot\mathbf{p} + icD + \frac{ac}{2}[D, \mathbf{p}] + \dots\right).$$

Finally, use it on the product to get

$$e^{i\mathbf{a}\cdot\mathbf{p}} e^{icD} e^{-i\mathbf{a}\cdot\mathbf{p}} e^{-icD} = \exp(ac[D, \mathbf{p}] + \dots).$$

Now, expand out the exponential to get the answer:

$$\mathbb{1} + ac[D, \mathbf{p}].$$

The same argument works in 3 dimensions by linearity, so we're done.

b) Note that the operation on the space that corresponds to the transformation given in (a) is:

$$\mathbf{x} \mapsto e^c \mathbf{x} \mapsto e^c \mathbf{x} + \mathbf{a} \mapsto \mathbf{x} + e^{-c} \mathbf{a} \mapsto \mathbf{x} + (e^{-c} - 1)\mathbf{a}.$$

Expanding the final term out to second order (since we went to second order in part (a)), we get

$$\mathbf{x} \mapsto \mathbf{x} - c\mathbf{a}.$$

By (a), the infinitesimal transformation which corresponds to this is exactly $\mathbb{1} + c\mathbf{a} \cdot [D, \mathbf{p}]$. Since \mathbf{p} is the generator of translations, we see that in order to generate a translation $\mathbf{x} \mapsto \mathbf{x} - c\mathbf{a}$, we need to take $[D, \mathbf{p}] = -\mathbf{p}$. If we had defined D to be the generator such that e^{-cD} is the corresponding unitary operator, then we get instead

$$[D, \mathbf{p}] = i\mathbf{p},$$

and this is how it's usually done in conformal field theory.