Review Session Problems 2

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Exercise 1. When we solve the hydrogen atom, we assume that the nucleus is a point charge. In this problem, we will compute the approximate change to the energy levels due to the finite size of the nucleus. This is called the **volume effect**. Model the nucleus as a uniform sphere of radius $r_0 A^{1/3}$, where $A^{1/3}$ is the number of nucleons (so this works for e.g. deuterium) and $r_0 = 1.3 \cdot 10^{-13}$ cm.

a) What is the potential $V(r)$?
Hint: Outside the nucleus, $V(r)$ is just the Coulomb potential. Inside the nucleus, use Gauss' law to determine $V(r)$.

b) What is H', where H^0 is the hydrogen atom hamiltonian?

c) Argue that the $\ell = 0$ states are only slightly affected by this perturbation.
Hint: Think about the small r behavior of the wavefunctions for s-states vs. $\ell > 0$ states.

d) Calculate the correction to the energy levels for all states with $\ell = 0$. Note that

$$R_{n0}(0) = \frac{2}{(na_0)^{3/2}},$$

where $a_0 = \hbar^2/me^2$.

e) For hydrogen, calculate the correction to the $n = 1$ and $n = 2$ states in eV.

f) Fine structure is of order $\alpha^4 mc^2$. Compare the magnitude of the volume effect to that of fine structure.

Exercise 2. Explain the physical origins and give the relative magnitudes of

a) fine structure

b) Lamb shift

c) hyperfine structure.

Exercise 3. ***Griffiths 8.19.*** Find the lowest bound on the ground state of hydrogen using the variational principle and an exponential trial wavefunction,

$$\psi(\mathbf{r}) = Ae^{-br^2},$$

where A is determined by normalization and b is a variational parameter. Express your answer in eV.

Exercise 4. ***Griffiths 9.18.*** When we turn on an external electric field, it should be possible to ionize the electron in an atom. A crude model for this is to suppose that a particle is in a very deep, one-dimensional finite square well from $x = -a$ to $x = a$.

a) What is the energy of the ground state, measured up from the bottom of the well? Assume that $V_0 \gg \hbar^2/ma^2$.

b) Introduce the perturbation $H' = -\alpha x$, where $\alpha \equiv eE_{\text{ext}}$. Assume that $\alpha a \ll \hbar^2/ma^2$, and sketch the total potential, noting that the electron can tunnel out in the direction of positive x.

c) Calculate

$$\gamma = \frac{1}{\hbar} \int |p(x)| \, \mathrm{d}x,$$

and estimate the time it would take for the particle to escape,

$$\tau = \frac{2x_1}{v} e^{2\gamma},$$

where x_1 is the distance the electron must travel to reach the tipping point of the potential and v is the speed of the electron.

d) Plug in some numbers, e.g. $V_0 = 20$ eV, $E_{\text{ext}} = 7 \cdot 10^6$ V/m, $a = 10^{-10}$ m. Calculate τ, and compare it to the age of the universe.