Final Review Session Problems

Jacob Erlikhman

December 5, 2024

Exercise 1. The integral form of the Schrödinger equation reads

$$\psi(\mathbf{r}) = \psi_0(\mathbf{r}) + \int G(\mathbf{r} - \mathbf{r}')V(\mathbf{r}')\psi(\mathbf{r}')\,\mathrm{d}^3\mathbf{r}',$$

where

$$G(\mathbf{r}) = -\frac{m}{2\pi\hbar^2} \cdot \frac{e^{ikr}}{r}$$

is the Green's function for the Schrödinger equation.

a) Use the method of successive approximations to write $\psi(\mathbf{r})$ as a series in the incident wavefunction $\psi_0(\mathbf{r})$.

b) Truncate the Born series you obtain after the second term to get the first Born approximation. Assuming the potential is localized near $\mathbf{r}' = 0$, we can write

$$\frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r} - \mathbf{r}'|} \approx \frac{e^{ikr}}{r} e^{-i\mathbf{k}\cdot\mathbf{r}'}.$$

Using this and the definition of $f(\theta)$,

$$\psi(\mathbf{r}) = Ae^{ikz} + f(\theta)\frac{e^{ikr}}{r},$$

determine $f(\theta)$.

c) In Griffiths, we find that for a potential $V(r) = V_0/r$, $f_{\text{point}}(\theta) = -\frac{2mV_0}{\hbar^2 q^2}$, where $\mathbf{q} = \mathbf{k}' - \mathbf{k}$. If $V(\mathbf{r}) = -e^2 Z/r$ for an electron scattering off a point charge of charge Ze, how would $f(\theta)$ change if instead the electron scatters off a spherical nucleus of radius a, charge Ze, and uniform charge density? Your answer should be of the form

$$f(\theta) = f_{\text{point}}(\theta) \cdot F(q),$$

where $F(q)$ is the **form factor** of the nucleus.

d) If you haven't done so already, calculate $F(q)$ explicitly.

e) From scattering high-energy electrons at nuclei, the actual form factor is measured to be

$$F(q) = \frac{Ze}{(1 + q^2 {a_N}^2)^2},$$

where $a_N \approx 0.26$ fm. If the inverse Fourier transform of $\frac{1}{(1+x^2)^2}$ is $e^{-|x|}$, what does that tell you about the size and charge density of the proton?

Exercise 2. Consider a 1D harmonic oscillator of angular frequency ω_0 that is perturbed by a time-dependent potential $V(t) = bx\cos(\omega t)$, where x is the displacement of the oscillator from equilibrium. Evaluate $\langle x \rangle$ by time-dependent perturbation theory. Discuss the validity of the result for $\omega \approx \omega_0$ and ω far from ω_0. *Hints*: You will need to use time-dependent perturbation theory as developed in the Week 14 Worksheet. This problem is too difficult to solve in full generality, so don't try to do that. Instead, try to consider special cases which elucidate all the physics but don't make the algebra too complicated. For example, you might want to first consider the case that $|\psi(0)\rangle$ is a single eigenstate of the unperturbed hamiltonian (this will be too simple). Then, consider upgrading this to more complicated linear combinations, and conjecture what the physics will be in the most general case using the previous results.

Exercise 3. *Griffiths 11.33* The spontaneous emission of the 21-cm hyperfine line in hydrogen is a magnetic dipole transition with rate

$$\Gamma = \frac{\omega^3}{3\pi\varepsilon_0\hbar c^3}\left|\left\langle B\left|\frac{\boldsymbol{\mu}_e + \boldsymbol{\mu}_p}{c}\right|A\right\rangle\right|^2,$$

where

$$\boldsymbol{\mu}_e = -\frac{e}{m_e}\mathbf{S}_e$$
$$\boldsymbol{\mu}_p = \frac{5.59e}{2m_p}\mathbf{S}_p.$$

On the Week 13 Worksheet, you showed the triplet has slightly higher energy than the singlet. Calculate (approximately) the lifetime of this transition.

Exercise 4. Consider a dynamical variable ξ that can take only two values, 1 or -1 (for example, σ_z is such an operator for a spin 1/2 particle). Denote the corresponding eigenvectors as $|+\rangle$ and $|-\rangle$. Now, consider the following states.

i) The one-parameter family of pure states

$$|\theta\rangle = \frac{1}{\sqrt{2}}(|+\rangle + e^{i\theta}|-\rangle)$$

for any real θ.

ii) The nonpure state

$$\rho = \frac{1}{2}(|+\rangle\langle+| + |-\rangle\langle-|).$$

Show that $\langle \xi \rangle = 0$ in all of these states. What, if any, are the physical differences between these various states, and how could they be measured?

Exercise 5. In the homework, you showed that the most general density matrix for a spin 1/2 particle is $\rho = \frac{1}{2}(1 + \mathbf{a} \cdot \boldsymbol{\sigma})$, where $\mathbf{a}$ is some 3-vector. If the system has a magnetic moment $\boldsymbol{\mu} = \frac{1}{2}\gamma\hbar\boldsymbol{\sigma}$ and is in a constant magnetic field $\mathbf{B}$, calculate $\rho(t)$. Describe the result geometrically in terms of the variation of the vector $\mathbf{a}$.
Hint: You will at some point arrive at a vector differential equation. It will be helpful at this point to recall how to solve the equation of motion of a charged particle in a magnetic field,

$$q\mathbf{v} \times \mathbf{B} = m\dot{\mathbf{v}}.$$