Week 11 Worksheet Solutions
Bouncing Ball

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Exercise 1. A ball of mass m bounces elastically on the floor.

a) What is the potential as a function of the height x above the floor?

b) Solve the Schrödinger equation. You don't need to normalize your solution.

   Hint: You should get Airy's differential equation, $\psi''(z) - z\psi(z) = 0$. One way to manipulate the Schrödinger equation into such a form is to notice that for $\psi''(x) - \alpha^3 x\psi(x) = 0$, $z = \alpha x$ does the trick. The solutions of this equation are the Airy functions, $\mathrm{Ai}(z)$ and $\mathrm{Bi}(z)$. The graphs of these functions are below.

c) Calculate (approximately) the first 4 energies, using $g = 10\ \mathrm{m/s^2}$ and $m = 0.100$ kg.

d) Now, analyze this problem using the WKB approximation. Find the allowed energies E_n in terms of m, g, and $\hbar$.

   Hint: The connecting WKB wavefunctions are

$$\psi(x) = \begin{cases} \dfrac{2D}{\sqrt{p(x)}} \sin\left(\dfrac{1}{\hbar}\displaystyle\int_x^{x_2} p(x')\,\mathrm{d}x' + \dfrac{\pi}{4}\right), & x < x_2 \\ \dfrac{D}{\sqrt{p(x)}} \exp\left(-\dfrac{1}{\hbar}\displaystyle\int_{x_2}^{x} |p(x')|\,\mathrm{d}x'\right), & x > x_2 \end{cases}.$$

e) Plug in the values from (c), and compare the WKB calculation to the "exact" one for the first four energies.

f) How large would n have to be to give the ball an average height of 1 meter above the ground?

a) We need to be careful here. The ball cannot go below the floor ($x = 0$), so the potential must be

$$V(x) = \begin{cases} mgx, & x \geq 0 \\ \infty, & x < 0 \end{cases}.$$

b) The Schrödinger equation is

$$-\frac{\hbar^2}{2m}\psi''(x) + mgx\psi(x) = E\psi(x).$$

Rewrite this as

$$\psi''(x) - \frac{2m^2 g}{\hbar^2}(x - E/mg)\psi(x) = 0.$$

Now, make the substitution

$$z = \alpha(x - E/mg),$$

where

$$\alpha \equiv \sqrt[3]{\frac{2m^2 g}{\hbar^2}},$$

so that the Schrödinger equation becomes

$$\psi''(z) - z\psi(z) = 0.$$

Note that we can freely replace the $z/\alpha + E/mg$ in the argument of ψ by z, since we are just looking for the solutions ψ. If we solve the ODE with the new argument z, the solution will already account for our replacement. More formally, we could define a function $f(z) = z/\alpha + E/mg$, and our solutions will not be solving for ψ but for $\varphi = \psi \circ f$. By abuse of notation, we still call φ by ψ. Now, the solutions of this equation will be Airy functions:

$$\psi(x) = C_1 \operatorname{Ai}(z) + C_2 \operatorname{Bi}(z),$$

where $C_i \in \mathbb{C}$ are constants. Now, use the boundary conditions. One boundary condition is given by the floor. Since $V(x) = \infty$ for $x < 0$, and since the wavefunction must be continuous at 0, it follows that $\psi(x = 0) = 0$. For a classical bouncing ball dropped from a height h, the maximum x value it can take is h. However, since the potential is finite for $x > h$, a quantum bouncing ball can tunnel all the way to ∞; hence, x can take any positive value. Looking at the graphs of the Airy functions, Bi diverges as $z \to \infty$, i.e. as $x \to \infty$, so that $C_2 = 0$. Thus, the solution is

$$\psi(x) = C_1 \operatorname{Ai}(\alpha x - \alpha E/mg).$$

c) On the other hand, the first boundary condition gives that $\psi(x = 0) = 0$; hence,

$$\operatorname{Ai}(-\alpha E/mg) = 0.$$

Again, we look at the graph of $\operatorname{Ai}(z)$, where we can see that $-\alpha E/mg$ must be a zero of Ai, of which there are countably many. Denoting these zeroes by $-z_n$ (so that $z_n > 0$), we have

$$\begin{aligned} E_n =& \frac{mgz_n}{\alpha} \\ =& z_n \sqrt[3]{\frac{mg^2\hbar^2}{2}} \end{aligned}$$

are the quantized energies of the bouncing ball.

We now calculate (all energies are in Joules)

$$\begin{aligned} E_1 &= 8.6 \cdot 10^{-23} \\ E_2 &= 1.5 \cdot 10^{-22} \\ E_3 &= 2.0 \cdot 10^{-22} \\ E_4 &= 2.5 \cdot 10^{-22}. \end{aligned}$$

d) After all that work, we now get to the easy part. The boundary condition $\psi(0) = 0$ gives us the quantization condition

$$\int_0^{x_2} p(x')\,\mathrm{d}x' = \pi\hbar(n - 1/4),$$

where x_2 is set by the energy, i.e. $x_2 = E/mg$. Likewise, $p(x) = \sqrt{2mE - 2m^2gx}$, so that

$$\begin{aligned} &(2mE - 2m^2gx)^{3/2}\Big|_0^{E/mg}\left(-\frac{1}{3m^2g}\right) = \pi\hbar(n - \pi/4) \Longrightarrow \\ \Longrightarrow &(2mE)^{3/2} = 3m^2g\pi\hbar(n - \pi/4) \Longrightarrow \\ \Longrightarrow &E_n = \frac{\left(3m^2g\pi\hbar(n - \pi/4)\right)^{2/3}}{2m}. \end{aligned}$$

e) Again, we calculate (in Joules)

$$\begin{aligned} E_1 &= 8.5 \cdot 10^{-23} \\ E_2 &= 1.2 \cdot 10^{-21} \\ E_3 &= 1.8 \cdot 10^{-21} \\ E_4 &= 2.5 \cdot 10^{-22}. \end{aligned}$$

These are within an order of magnitude to the exact solutions, which is still really close for the amount of work necessary!

f) $E/mg = 1$ m, so $E = 1$ J. In order to get the ball that high up, we'd have to take

$$\begin{aligned} n &= \frac{(2m)^{3/2}}{3m^2g\pi\hbar} + \pi/4 \\ &= 9.5 \cdot 10^{32}! \end{aligned}$$

Notice that we plugged in the maximum height as 1 m, when we were supposed to assume an *average* height of 1 m. But a bouncing ball will have an average height that is near the maximum (since it spends most of its time there), and the true average will have an n value different from 10^{33} by a relatively small number that would be impossible to measure. So we can take this n for the average height as well.