Week 3 Worksheet
Identical Particles

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Exercise 1. ***Griffiths 5.5.***

a) Write down the hamiltonian for two noninteracting identical particles in the infinite square well. Write down the ground states for the three cases: distinguishable, fermions, bosons. Recall that the one-particle wavefunctions are

$$\psi_n(x) = \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi}{a}x\right),$$

with energies $E_n = n^2\pi^2\hbar^2/2ma^2$.

b) Find the first three excited states and their energies for each of the three cases (distinguishable, fermions, bosons).

Exercise 2. ***Griffiths 5.9.*** In Exercise 1, we ignored spin (or at least supposed that the particles are in the same spin state).

a) Do it now for particles of spin 1/2. Construct the four lowest-energy configurations, and specify their energies and degeneracies.
Hint: Recall that the *total* state vector for a boson (resp. fermion) must be symmetric (resp. anti-symmetric). If a boson or fermion state vector is a product of two vectors (e.g. a spatial state vector and a spin state vector), can these components be symmetric, anti-symmetric, or both?

b) Do the same for spin 1.
Hint: You can do this without having to use any Clebsh-Gordan coefficients!

Exercise 3. Symmetries of Many-Particle States.

a) Consider a system of two identical particles. Define the operator P_{12} via

$$P_{12} \left|a\right\rangle \left|b\right\rangle = \left|b\right\rangle \left|a\right\rangle.$$

Show that ${P_{12}}^2 = \mathbb{1}$, the identity operator, and that the eigenvalues of P_{12} are ± 1. Thus, show that its eigenvectors are either totally symmetric or antisymmetric. We call P_{12} a **permutation operator**. In this case, there are only two such operators: P_{12} and ${P_{12}}^2 = \mathbb{1}$.

b) Generalize part (a) to systems of three identical particles. You should find that you have *six* permutation operators (note that the identity is a permutation operator). Assuming the hamiltonian is invariant under each of these operators, is there a complete set of common eigenvectors?

c) ***Griffiths 5.8.*** In the situation of (b), suppose that the particles have access to three distinct one-particle states, $|a\rangle$, $|b\rangle$, and $|c\rangle$. For example, $|abc\rangle$ is an allowed state, as is $|aaa\rangle$. How many states can be constructed if they are (i) bosons or (ii) fermions?

d) Suppose we have a single-particle fermion state $|\alpha\rangle$ and a single-particle bosonic state $|\beta\rangle$. Just like for the harmonic oscillator, we can define **creation operators** $C_\alpha{}^\dagger$ and $a_\beta{}^\dagger$, such that given any state $|\psi\rangle$,

$$\begin{aligned} C_\alpha{}^\dagger |\psi\rangle &= |\alpha\psi\rangle \\ a_\beta{}^\dagger |\psi\rangle &= |\beta\psi\rangle . \end{aligned}$$

The operators $C_\alpha{}^\dagger$ and $a_\beta{}^\dagger$ have the following properties. You don't need to prove them.

$$\begin{aligned} C_\alpha |\alpha\psi\rangle &= |\psi\rangle \\ a_\beta |\beta\psi\rangle &= |\psi\rangle \\ C_\alpha |0\rangle = a_\beta |0\rangle &= 0 \\ C_\alpha{}^\dagger C_\alpha{}^\dagger &= 0 \\ \{C_\alpha, C_{\alpha'}{}^\dagger\} \equiv C_\alpha C_{\alpha'}{}^\dagger + C_{\alpha'}{}^\dagger C_\alpha &= \delta_{\alpha\alpha'}\mathbb{1} \\ \{C_\alpha{}^\dagger, C_{\alpha'}{}^\dagger\} &= 0 \\ [a_\beta, a_{\beta'}{}^\dagger] &= \delta_{\beta\beta'}\mathbb{1} \\ [a_\beta{}^\dagger, a_{\beta'}{}^\dagger] &= 0, \end{aligned}$$

where $|0\rangle$ denotes a state with no particles at all. To what extent is a bound pair of fermions equivalent to a boson?
Hint: Use the symmetries of many-particle states and the (anti-)commutation relations of the creation/annihilation operators constructed in parts (a)-(d). What algebra must the creation/annihilation operators for the bound pair satisfy?

e) **Challenge.** Prove the properties given in (d).
Hints: It may be useful to use the notation $\sim\alpha$ for the α "orbital" being *unoccupied*. To show the first relation for C_α, try to first show that $C_\alpha |\alpha\rangle = |0\rangle$. For the anti-commutator relations, consider separately the cases $\alpha \neq \alpha'$ and whether the α or α' orbitals are occupied.