Week 4 Worksheet
Free Electron Gas

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Exercise 1. Suppose you have N electrons in a box of side length L.

a) Show that the Fermi energy is

$$E_F = \frac{\hbar^2}{2m}\left(\frac{3N\pi^2}{L^3}\right)^{2/3}.$$

b) Find the total energy of the electrons in terms of E_F.

This is in Griffiths.

Exercise 2. Now, consider a free electron gas in two dimensions, confined to a square of side length L.

a) ***Griffiths 5.30.*** Find the Fermi energy in terms of N and L, and show that the average energy of the particles is $E_F/2$.

b) Let $g(E)dE$ be the number of particles with energy E in the interval dE. $g(E)$ is called the **density of states** and is useful in various problems in quantum statistical mechanics. Calculate $g(E)$ for the particles. Your formula should be constant, i.e. independent of E.

a) We want to calculate the integral over a quarter disk, which, after we integrate out the angular part of the integral, is

$$\frac{1}{4}\cdot 2\pi\int_0^{n_{\max}} \mathrm{d}n\, n = N/2,$$

since we have 2 electrons in each energy level (here, $n^2 = {n_x}^2 + {n_y}^2$). For electrons in a box, the energy is

$$E_n = \frac{n^2\pi^2\hbar^2}{2mL^2}.$$

Thus,

$${n_{\max}}^2 = \frac{2mL^2E_F}{\hbar^2\pi^2}.$$

Plugging this into the integral and solving for E_F, we get the answer:

$$E_F = \frac{N\hbar^2\pi}{mL^2}.$$

The total energy is given by

$$E_{\text{tot}} = 2 \cdot \frac{2\pi}{4} \int_0^{n_{\max}} \mathrm{d}n\, n\, E_n.$$

We get the factor of 2 because we have two electrons at each energy level, the factor of 1/4 because we are only integrating over a quarter disk, and the factor of 2π because we integrated out the angular part (remember that $\mathrm{d}n_x\, \mathrm{d}n_y = n\, \mathrm{d}n\, \mathrm{d}\theta$). Plugging everything in and noting that ${n_{\max}}^2 = 2N/\pi$, we get that $E_{\text{tot}} = NE_F/2$, from which it follows that the average energy is $E_F/2$.

b) Observe that

$$2\int_0^{E_{\max}} g(E)\, \mathrm{d}E = N.$$

From (a), we also have that

$$N = \pi \int_0^{n_{\max}} \mathrm{d}n\, n.$$

Now,

$$n = \frac{L\sqrt{2mE}}{\pi\hbar} \Longrightarrow \mathrm{d}n = \frac{L}{\pi\hbar}\sqrt{\frac{m}{2E}}\, \mathrm{d}E.$$

Combining the above 3 equations, we get

$$2\int g(E)\, \mathrm{d}E = \pi \int \frac{L^2}{\hbar^2}\frac{m}{\pi^2}\, \mathrm{d}E.$$

Thus,

$$g(E) = \frac{mL^2}{2\pi\hbar^2}.$$