

Week 5 Worksheet

Symmetries!

Jacob Erlikhman

September 26, 2024

Exercise 1. In this problem, you will construct the 2×2 matrix corresponding to a finite rotation which places the \hat{z} axis along an arbitrary direction \hat{r} .

- a) A rotation can be specified by the Euler angles (α, β, γ) or by rotations about the three axes (x, y, z) by angles (η, θ, φ) ; note that you always need 3 angles to specify a general rotation. The Euler angles represent first a rotation about \hat{z} by an angle α , then a rotation *about the new y-axis* by an angle β , and then a rotation about the *new z-axis* again. Convince yourself that this works.
- b) Now, suppose given a rotation specified by the Euler angles (α, β, γ) . This is given in quantum mechanics by the matrix

$$e^{-i\gamma S_{z'}/\hbar} e^{-i\beta S_u/\hbar} e^{-i\alpha S_z/\hbar},$$

where the u -axis is the new y -axis after rotating about z , and the z' -axis is the new z -axis after rotating about \hat{z} and \hat{u} . Show that this is the same matrix as

$$e^{-i\alpha S_z/\hbar} e^{-i\beta S_y/\hbar} e^{-i\gamma S_z/\hbar}.$$

Hint: Denoting a rotation about the axis r by an angle ζ as $R_r(\zeta)$, we have that $S_u = R_z(\alpha)S_yR_z(-\alpha) = e^{-i\alpha S_z/\hbar}S_ye^{i\alpha S_z/\hbar}$. Using the $R_r(\zeta)$ notation throughout, try to write $R_u(\beta)$, $R_{z'}(\gamma)$ in terms of R_z 's and R_y 's.

- c) Use part (b) with $S_i = \frac{\hbar}{2}\sigma_i$ to calculate the rotation matrix corresponding to placing the \hat{z} axis along \hat{r} , where \hat{r} is specified by the two angles (θ, φ) .
Hints: The idea is to Taylor expand each exponential. Think about a simple expression for σ_i^n , where σ_i is the Pauli matrix you need. Finally, one of the results you should get along the way is

$$e^{-i\beta\sigma_y/2} = \cos(\beta/2)\mathbb{1} - i\sigma_y \sin(\beta/2).$$

- d) **Griffiths 6.32(f)**. Calculate the matrix corresponding to a rotation by π about \hat{x} .
- e) **Griffiths 6.32(g)**. Calculate the matrix corresponding to a 2π rotation about \hat{z} . Comment on the answer.

Exercise 2. Another symmetry is called **dilation** symmetry. Dilations are given by the transformation $\mathbf{x} \rightarrow \mathbf{x}' = e^c \mathbf{x}$, where $c \in \mathbb{R}$. Call its generator D , so that e^{-icD} is the corresponding unitary operator.

Remark. In conformal field theory, the convention is to absorb the factor of i into D , so that e^{-cD} is the dilation operator.

- a) Show that the *infinitesimal* transformation

$$e^{i\mathbf{a}\cdot\mathbf{p}} e^{icD} e^{-i\mathbf{a}\cdot\mathbf{p}} e^{-icD}$$

is given by $\mathbb{1} + c\mathbf{a} \cdot [D, \mathbf{p}]$.

Hints: You can reduce to the situation where all the vectors are 1-dimensional (why?). There's a slick way to solve this problem (use the Baker-Campbell-Hausdorff formula), but the brute force method does work.

- b) Calculate $[D, \mathbf{p}]$.

Hint: What coordinate transformation does the above correspond to? In other words, if you write it in the form $\mathbf{x} \rightarrow \mathbf{x}'$, what is \mathbf{x}' ?