Week 8 Worksheet
(Nondegenerate) Peturbation Theory

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Exercise 1. Let $H = H^0 + H^1$ be a perturbed hamiltonian. Suppose we know

$$H^0 |n^0\rangle = E_n^0 |n^0\rangle ,$$

where the $|n^0\rangle$ are the unperturbed, orthonormal, nondegenerate eigenstates.

a) Expand the exact solutions for H, $|n\rangle$ and E_n, in perturbation expansions.

b) Write the Schrödinger equation for H in terms of the above expansions.

c) Study the first order part of the equation from (b), and derive the first order corrections to the energies. You should get

$$E_n^1 = \langle n^0 | H^1 | n^0\rangle .$$

d) Along the way to solving (c), you should have come up with the equation

$$H^0 |n^1\rangle + H^1 |n^0\rangle = E_n^1 |n^0\rangle + E_n^0 |n^1\rangle .$$

Using this equation, find the expansion of $|n^1\rangle$ in the eigenbasis of H^0.
Hints: In order to find the component of $|n^1\rangle$ which is parallel to $|n^0\rangle$, enforce normalization of $|n\rangle$ to first order, i.e. $|n^0\rangle + |n^1\rangle$ should have norm 1. It will be helpful to write $|n^1\rangle = |n_\parallel\rangle + |n_\perp\rangle$, where $\langle n^0 | n_\perp\rangle = 0$; also, use the fact that—to first order—$e^{ia} = 1 + ia$.

e) Derive the second order corrections to the energies, E_n^2.

Exercise 2. Suppose you want to calculate the expectation value of some observable A in the n^{th} energy eigenstate of a system perturbed by H^1,

$$\langle A\rangle = \langle n | A | n\rangle .$$

Suppose further that all eigenstates are nondegenerate.

a) Replace $|n\rangle$ by its perturbation expansion, and write down the formula for the first order correction to $\langle A\rangle$, $\langle A\rangle^1$.

b) Use the first order corrections to the states,

$$|n^1\rangle = \sum_{m\neq n} \frac{\langle m^0|\, H^1\, |n^0\rangle}{E_n^0 - E_m^0}\, |m^0\rangle\,,$$

to rewrite $\langle A\rangle^1$ in terms of the unperturbed eigenstates.

c) If $A = H^1$, what does the result of (b) tell you? Explain why this is consistent with the result of Exercise 1(e),

$$E_n^2 = \sum_{m\neq n} \frac{|\,\langle m|H^1|n\rangle\,|^2}{E_n^0 - E_m^0}.$$