# Final Review Session Solutions

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**Exercise 1.** ***Griffiths 8.19.*** Find the lowest bound on the ground state of hydrogen using the variational principle and an exponential trial wavefunction,

$$\psi(\mathbf{r}) = Ae^{-br^2},$$

where $A$ is determined by normalization and $b$ is a variational parameter. Express your answer in eV.

First, we calculate $A$. We can totally ignore the angular part of the integration, since any integration constant can be absorbed into $A$ anyway. We thus get

$$\int_0^\infty |A|^2 e^{-2br^2} r^2 \, \mathrm{d}r = |A|^2 \frac{\sqrt{\pi}}{4(2b)^{3/2}},$$

so

$$|A|^2 = \frac{4(2b)^{3/2}}{\sqrt{\pi}}.$$

The hydrogen atom hamiltonian is

$$H = -\frac{\hbar^2}{2m}\nabla^2 - \frac{e^2}{4\pi\varepsilon_0 r}.$$

Note that it's important that we have the laplacian here, so the derivative with respect to $r$ is not just $\partial^2/\partial r^2$. Acting on $\psi(r)$ with this hamiltonian, we get

$$-\frac{\hbar^2}{2m} A \left(2br^2 - 3\right) 2be^{-br^2} - \frac{e^2}{4\pi\varepsilon_0 r} Ae^{-br^2}.$$

We want to calculate $\langle\psi|H|\psi\rangle$ to use the variational principle. We thus calculate

$$\langle\psi|H|\psi\rangle = -\frac{\hbar^2}{2m}|A|^2 \int_0^\infty 2br^2 e^{-2br^2}(2br^2 - 3)\, \mathrm{d}r - \frac{e^2}{4\pi\varepsilon_0}|A|^2 \int_0^\infty re^{-2br^2}\, \mathrm{d}r.$$

The first integral has two terms, each of which is a gaussian integral. You can look up how to do these online or just plug them into e.g. Mathematica.

Note that gaussian integrals follow from the following calculation. Let

$$I = \int_{-\infty}^{\infty} \mathrm{d}x e^{-ax^2}.$$

First, we can do a change of variables $x \to x/\sqrt{a}$, so that

$$I = \frac{1}{\sqrt{a}} \int_{-\infty}^{\infty} \mathrm{d}x e^{-x^2}.$$

Then consider

$$I^2 = \frac{1}{a} \int_{-\infty}^{\infty} \mathrm{d}x e^{-(x^2+y^2)}.$$

We can set $r^2 = x^2 + y^2$, change to polar coordinates, and find

$$I^2 = \frac{2\pi}{a} \cdot \frac{I}{2},$$

where we get $I/2$ because the integration bounds for $r$ are from 0 to $\infty$. Since $I$ converges and is nonzero, we have

$$I = \sqrt{\frac{\pi}{a}}.$$

If we want to calculate integrals of the form

$$\int_0^{\infty} x^{2n} e^{-ax^2}\,\mathrm{d}x,$$

we can differentiate our form for $I$ with respect to $a$:

$$\frac{\mathrm{d}}{\mathrm{d}a} I = -\int \mathrm{d}x\ x^2 e^{-ax^2} = -\frac{\sqrt{\pi}}{2a^{3/2}}.$$

Similarly, differentiating twice with respect to $a$, we get

$$\frac{\mathrm{d}^2}{\mathrm{d}a^2} I = \int_{-\infty}^{\infty} \mathrm{d}x\ x^4 e^{-ax^2} = \frac{3\sqrt{\pi}}{4a^{5/2}}.$$

Returning to our calculation, we see that the second integral is easy, since the derivative of $e^{-2br^2}$ is $-4bre^{-2br^2}$, while the other two integrals are given by the calculations above. The result is

$$\begin{aligned}
\langle\psi|H|\psi\rangle &= -\frac{\hbar^2}{2m} \cdot \frac{4(2b)^{3/2}}{\sqrt{\pi}} \cdot 2b \left( \frac{3\sqrt{\pi}}{8(2b)^{5/2}} \cdot 2b - \frac{3\sqrt{\pi}}{4(2b)^{3/2}} \right) - \frac{e^2}{4\pi\varepsilon_0} \cdot \frac{4(2b)^{3/2}}{\sqrt{\pi}} \cdot \frac{1}{4b} \\
&= -\frac{\hbar^2}{2m} 2b \left( \frac{3}{2} - 3 \right) - \frac{e^2}{4\pi\varepsilon_0} \cdot \sqrt{\frac{8b}{\pi}} \\
&= \frac{3\hbar^2}{2m} b - \frac{e^2}{2\pi^{3/2}\varepsilon_0} \sqrt{2b}.
\end{aligned}$$

Now, we want to minimize $\langle\psi|H|\psi\rangle$ to get the best possible bound on the ground state energy. Hence, take the derivative with respect to $b$ and set it equal to 0, so that

$$\frac{3\hbar^2}{2m} = \frac{e^2\sqrt{2}}{4\pi^{3/2}\varepsilon_0\sqrt{b}} \Longrightarrow$$
$$\Longrightarrow b = \left(\frac{me^2\sqrt{2}}{6\pi^{3/2}\hbar^2\varepsilon_0}\right)^2.$$

Plugging this back in, we find

$$\langle\psi|H|\psi\rangle = -\frac{me^4}{12\pi^3\hbar^2\varepsilon_0^2}.$$

Plugging in some numbers, we get

$$E_g \leq -11.66 \text{ eV},$$

which is spectacularly close to the actual answer $-13.6$ eV.

**Exercise 2.** Consider a 1D harmonic oscillator of angular frequency $\omega_0$ that is perturbed by a time-dependent potential $V(t) = bx\cos(\omega t)$, where $x$ is the displacement of the oscillator from equilibrium. Evaluate $\langle x\rangle$ by time-dependent perturbation theory. Do not assume that the initial state is an eigenstate of the unperturbed system (it could be a linear combination of such eigenstates). Discuss the validity of the result for $\omega \approx \omega_0$ and $\omega$ far from $\omega_0$.
*Hints*: You will need to use time-dependent perturbation theory as developed in the Week 9 Worksheet. This problem is too difficult to solve in full generality, so don't try to do that. Instead, try to consider special cases which elucidate all the physics but don't make the algebra too complicated. For example, you might want to first consider the case that $|\psi(0)\rangle$ is a single eigenstate of the unperturbed hamiltonian. Then, consider upgrading this to more complicated linear combinations, and conjecture what the physics will be in the most general case using the previous results.

First, let's figure out what $H'_{nm} = \langle n|V(t)|m\rangle$ is. This will be given by

$$H'_{nm} = b\cos(\omega t)\sqrt{\frac{\hbar}{2m\omega}}\left(\delta_{m,n+1}\sqrt{n+1} + \delta_{m,n-1}\sqrt{n}\right),$$

where we use the form for $x$ in terms of raising and lowering operators. Now, we know that in first order time-dependent perturbation theory,

$$\frac{\mathrm{d}c_n}{\mathrm{d}t} = -\frac{i}{\hbar}\sum H'_{nm}e^{i\omega_{nm}t}c_m,$$

where the $c_i$ are the coefficients of the wavefunction at time $t = 0$, i.e.

$$|\psi(0)\rangle = \sum_m c_m|m\rangle.$$

Plugging in our result for $H'_{nm}$, we get

$$\begin{aligned}\frac{\mathrm{d}c_n}{\mathrm{d}t} &= -\frac{ib\cos(\omega t)}{\sqrt{2m\hbar\omega_0}}\left(c_{n+1}e^{-i\omega_0 t}\sqrt{n+1}+c_{n-1}e^{i\omega_0 t}\sqrt{n}\right)\\ &= -\frac{ib}{\sqrt{2m\omega_0\hbar}}\left[\left(e^{i(\omega-\omega_0)t}+e^{-i(\omega+\omega_0)t}\right)c_{n+1}\sqrt{n+1}+\left(e^{i(\omega+\omega_0)t}+e^{-i(\omega-\omega_0)t}\right)c_{n-1}\sqrt{n}\right].\end{aligned}$$

Integrating this from 0 to $t$, we get

$$\begin{aligned}c_n(t) = -\frac{b}{\sqrt{2m\hbar\omega_0}}\Bigg[&\left(\frac{e^{i(\omega-\omega_0)t}}{\omega-\omega_0}-\frac{e^{-i(\omega+\omega_0)t}}{\omega+\omega_0}-\frac{2\omega_0}{\omega^2-{\omega_0}^2}\right)c_{n+1}\sqrt{n+1}+\\ &+\left(\frac{e^{i(\omega+\omega_0)t}}{\omega+\omega_0}-\frac{e^{-i(\omega-\omega_0)t}}{\omega-\omega_0}+\frac{2\omega_0}{\omega^2-{\omega_0}^2}\right)c_{n-1}\sqrt{n}\Bigg].\end{aligned}$$

This is the most general expression for the time-dependence of the coefficients, from which we could then derive the most general expression for the time-dependence of $\langle x\rangle$. Instead, let's consider some more simple cases. The simplest case is $|\psi(0)\rangle = |n\rangle$. Then only the $c_{n-1}$ and $c_{n+1}$ coefficients acquire a time dependence. However, let's consider the more general case of the form

$$|\psi(0)\rangle = \alpha|n\rangle + \beta|n+1\rangle.$$

Note that this subsumes the simpler case where $|\psi(0)\rangle$ is an eigenstate of the harmonic oscillator by setting $\beta = 0$. Let's figure out what will happen here. We then have

$$\begin{aligned}c_{n-1}(t) &= -\frac{b}{\sqrt{2m\hbar\omega_0}}\left(\frac{e^{i(\omega-\omega_0)t}}{\omega-\omega_0}-\frac{e^{-i(\omega+\omega_0)t}}{\omega+\omega_0}-\frac{2\omega_0}{\omega^2-{\omega_0}^2}\right)\alpha\sqrt{n}\\ c_n(t) &= -\frac{b}{\sqrt{2m\hbar\omega_0}}\left(\frac{e^{i(\omega-\omega_0)t}}{\omega-\omega_0}-\frac{e^{-i(\omega+\omega_0)t}}{\omega+\omega_0}-\frac{2\omega_0}{\omega^2-{\omega_0}^2}\right)\beta\sqrt{n+1}\\ c_{n+1}(t) &= -\frac{b}{\sqrt{2m\hbar\omega_0}}\left(\frac{e^{i(\omega+\omega_0)t}}{\omega+\omega_0}-\frac{e^{-i(\omega-\omega_0)t}}{\omega-\omega_0}+\frac{2\omega_0}{\omega^2-{\omega_0}^2}\right)\alpha\sqrt{n+1}\\ c_{n+2}(t) &= -\frac{b}{\sqrt{2m\hbar\omega_0}}\left(\frac{e^{i(\omega+\omega_0)t}}{\omega+\omega_0}-\frac{e^{-i(\omega-\omega_0)t}}{\omega-\omega_0}+\frac{2\omega_0}{\omega^2-{\omega_0}^2}\right)\beta\sqrt{n+2},\end{aligned}$$

with all other $c_i(t) = 0$. This means that

$$|\psi(t)\rangle = c_{n-1}(t)\,|n-1\rangle + c_n(t)|n\rangle + c_{n+1}(t)|n+1\rangle + c_{n+2}(t)\,|n+2\rangle\,.$$

Now, let's try to calculate $\langle x(t)\rangle$. For a general $|\psi\rangle = \sum c_n\,|n\rangle$, we have

$$\langle x\rangle = \sum_n \sqrt{\frac{\hbar}{2m\omega}}c_n^*\left(\sqrt{n+1}c_{n+1}+\sqrt{n}c_{n-1}\right).$$

Since our $|\psi\rangle$ only has 4 nonzero $c_i$,

$$\begin{aligned}\langle x\rangle =& \sqrt{\frac{\hbar}{2m\omega}}\Big[c_n^*\left(\sqrt{n+1}c_{n+1}+\sqrt{n}c_{n-1}\right)+c_{n-1}^*\sqrt{n}c_n+c_{n+1}^*\left(\sqrt{n+2}c_{n+2}+\sqrt{n+1}c_n\right)+c_{n+2}^*\sqrt{n+2}c_{n+1}\\ =&2\sqrt{\frac{\hbar}{2m\omega}}\left(\sqrt{n+1}\text{Re}\left[c_n^*c_{n+1}\right]+\sqrt{n}\text{Re}\left[c_n^*c_{n-1}\right]+\sqrt{n+2}\text{Re}\left[c_{n+2}^*c_{n+1}\right]\right).\end{aligned}$$

So we only need to calculate $c_n^*c_{n+1}$, $c_n^*c_{n-1}$, and $c_{n+2}^*c_{n+1}$. The first is given by (after a lot of algebra)

$$\begin{aligned}c_n^*c_{n+1} = \frac{\alpha\beta^*b^2}{2m\hbar\omega_0}(n+1)\Bigg(&\frac{2e^{2i\omega_0 t}}{\omega^2-\omega_0{}^2}-\frac{e^{-2i(\omega-\omega_0)t}}{(\omega-\omega_0)^2}-\frac{e^{2i(\omega+\omega_0)t}}{(\omega+\omega_0)^2}-\frac{4\omega_0 e^{i(\omega+\omega_0)t}}{(\omega+\omega_0)(\omega^2-\omega_0{}^2)}+\\ &+\frac{4\omega_0 e^{-i(\omega-\omega_0)t}}{(\omega-\omega_0)(\omega^2-\omega_0{}^2)}-\frac{4\omega_0{}^2}{(\omega^2-\omega_0{}^2)^2}\Bigg)\end{aligned}$$

The real part of this is then

$$\begin{aligned}\text{Re}\left[c_n^*c_{n+1}\right] = \frac{\text{Re}\left[\alpha\beta^*\right]b^2}{2m\hbar\omega_0}(n+1)\Bigg(&\frac{2\cos(2\omega t)}{\omega^2-\omega_0{}^2}-\frac{\cos[2(\omega-\omega_0)t]}{(\omega-\omega_0)^2}-\frac{\cos[2(\omega+\omega_0)t]}{(\omega+\omega_0)^2}-\\ &-\frac{4\omega_0\cos[(\omega+\omega_0)t]}{(\omega+\omega_0)(\omega^2-\omega_0{}^2)}+\frac{4\omega_0\cos[(\omega-\omega_0)t]}{(\omega-\omega_0)(\omega^2-\omega_0{}^2)}-\frac{4\omega_0{}^2}{(\omega^2-\omega_0{}^2)^2}\Bigg)\end{aligned}$$

Similarly, the second is

$$\begin{aligned}c_n^*c_{n-1} = \frac{b^2\beta^*\alpha}{2m\hbar\omega_0}\sqrt{(n+1)n}\Bigg(&\frac{1}{(\omega-\omega_0)^2}+\frac{4\omega_0{}^2}{(\omega^2-\omega_0{}^2)^2}+\frac{1}{(\omega+\omega_0)^2}-\frac{2\cos(2\omega t)}{\omega^2-\omega_0^2}-\\ &-\frac{4\omega_0\cos[(\omega-\omega_0)t]}{(\omega-\omega_0)(\omega^2-\omega_0{}^2)}+\frac{4\omega_0\cos[(\omega+\omega_0)t]}{(\omega+\omega_0)(\omega^2-\omega_0{}^2)}\Bigg).\end{aligned}$$

Lastly, we have

$$\begin{aligned}c_{n+2}^*c_{n+1} = \frac{b^2\beta^*\alpha}{2m\hbar\omega_0}\sqrt{(n+1)(n+2)}\Bigg(&\frac{1}{(\omega+\omega_0)^2}+\frac{1}{(\omega-\omega_0)^2}-\frac{2\cos(2\omega t)}{\omega^2-\omega_0{}^2}+\frac{4\omega_0\cos[(\omega+\omega_0)t]}{(\omega+\omega_0)(\omega^2-\omega_0{}^2)}-\\ &-\frac{4\omega_0\cos[(\omega-\omega_0)t]}{(\omega-\omega_0)(\omega^2-\omega_0{}^2)}+\frac{4\omega_0{}^2}{(\omega^2-\omega_0{}^2)^2}\Bigg)\end{aligned}$$

Note that the answer we have obtained is for a system in a mixing of two eigenmodes of the harmonic oscillator. This means that the initial system had two associated frequencies, $n\omega$ and $(n+1)\omega$, so it's not surprising that we have obtained an answer which causes the mixing to extend to nearby states with (at first glance) no patterns. However, we can notice that when we are near resonance, $\omega \sim \omega_0$, then the dominant terms in $\langle x\rangle$ are given by several frequencies: $2\omega$, $\omega+\omega_0$, and $2(\omega+\omega_0)$.

On the otherhand, if we instead suppose the initial state was *in a single energy eiegenstate*, $|\psi\rangle = |n\rangle$, then $c_{n-1}$ and $c_{n+1}$ are as in the previous case (with $\alpha$ set equal to 1), while $c_n = 1$ is constant for all time. This follows from the general case above by setting $\beta = 0$. Thus, we see that the dominant frequency near resonance is $\omega+\omega_0$, and the position of equilibrium is shifted from $x = 0$. On the other hand, when $\omega$

is far from $\omega_0$, we have two frequencies $\omega - \omega_0$ and $\omega + \omega_0$, which correspond to the eigenmodes of the classical driven harmonic oscillator! So this seems commensurate with the classical equation of motion. Note that this makes sense, because expectation values are expected to obey classical equations of motion.

Lastly, we could consider the more general case,

$$|\psi(0)\rangle = \alpha|n-1\rangle + \beta|n\rangle + \gamma|n+1\rangle,$$

but I don't think we're going to get any more physics out of it. I expect that for $\omega$ near $\omega_0$, we should end up with the same conclusions as in the analysis above for $\gamma = 0$, and similarly for $\omega$ far from $\omega_0$.

**Exercise 3.** ***Griffiths 11.33*** The spontaneous emission of the 21-cm hyperfine line in hydrogen is a magnetic dipole transition with rate

$$\Gamma = \frac{\omega^3}{3\pi\varepsilon_0\hbar c^3}\left|\left\langle B\left|\frac{\boldsymbol{\mu}_e + \boldsymbol{\mu}_p}{c}\right|A\right\rangle\right|^2,$$

where

$$\begin{aligned}\boldsymbol{\mu}_e &= -\frac{e}{m_e}\mathbf{S}_e\\ \boldsymbol{\mu}_p &= \frac{5.59e}{2m_p}\mathbf{S}_p.\end{aligned}$$

On the Week 11 Worksheet, you showed the triplet has slightly higher energy than the singlet. Calculate (approximately) the lifetime of this transition.

Call the triplet state(s) $|1\rangle$ and the singlet state $|0\rangle$. We then have (remembering that $m_p \gg m_e$)

$$\begin{aligned}\Gamma &\approx \frac{\omega^3 e^2}{3\pi\varepsilon_0\hbar c^5}\left|\left\langle 0\left|\frac{e^2}{{m_e}^2}\mathbf{S}_e\right|1\right\rangle\right|^2\\ &= \frac{\omega^3 e^4}{3\pi\varepsilon_0\hbar c^5 {m_e}^2}\left|\langle 0|\mathbf{S}_e|1\rangle\right|^2.\end{aligned}$$

Evaluating the matrix element, we get

$$\langle 0|\mathbf{S}_e|1\rangle = \frac{\hbar}{2\sqrt{2}}\Big[\big(\langle\uparrow\downarrow| - \langle\downarrow\uparrow|\big)\boldsymbol{\sigma}_e|\uparrow\uparrow\rangle\Big] = -\frac{\hbar}{2\sqrt{2}}(\hat{x} + i\hat{y}),$$

which you can obtain by using any triplet. Note that the first arrow is the electron and the second the proton. Thus,

$$\left|\langle 0|\mathbf{S}_e|1\rangle\right|^2 = \frac{\hbar^2}{4}.$$

Plugging this in to $\Gamma$, we get

$$\begin{aligned}\Gamma &= \frac{\omega^3 e^2}{3\pi\varepsilon_0\hbar c^5 {m_e}^2}\cdot\frac{\hbar^2}{4}\\ &= \alpha\frac{4\hbar^2\omega^3}{12c^4 {m_e}^2}.\end{aligned}$$

Since $\omega = 2\pi c/\lambda$, we can plug in and evaluate ($\lambda = 21$ cm). We find

$$\Gamma \approx 10^{-14}\ \mathrm{s}^{-1},$$

so $T \approx 10^{14}$ s or $10^7$ years.

**Exercise 4.** ***Griffiths 9.18.*** When we turn on an external electric field, it should be possible to ionize the electron in an atom. A crude model for this is to suppose that a particle is in a very deep, one-dimensional finite square well from $x = -a$ to $x = a$.

a) What is the energy of the ground state, measured up from the bottom of the well? Assume that $V_0 \gg \hbar^2/ma^2$.

b) Introduce the perturbation $H' = -\alpha x$, where $\alpha \equiv eE_{\mathrm{ext}}$. Assume that $\alpha a \ll \hbar^2/ma^2$, and sketch the total potential, noting that the electron can tunnel out in the direction of positive $x$.

c) Calculate

$$\gamma = \frac{1}{\hbar}\int |p(x)|\,\mathrm{d}x,$$

and estimate the time it would take for the particle to escape,

$$\tau = \frac{2x_1}{v}e^{2\gamma},$$

where $x_1$ is the distance the electron must travel to reach the tipping point of the potential and $v$ is the speed of the electron.

d) Plug in some numbers, e.g. $V_0 = 20$ eV, $E_{\mathrm{ext}} = 7 \cdot 10^6$ V/m, $a = 10^{-10}$ m. Calculate $\tau$, and compare it to the age of the universe.

a) In the limit $V_0 \gg \hbar^2/ma^2$, this is just the ground state energy of the *infinite* square well of width $2a$, which is

$$\frac{\hbar^2\pi^2}{8ma^2}.$$

b) The potential is

$$V(x) = \begin{cases} -\alpha x, & x \in (-a, a) \\ V_0 - \alpha x, & x > a \end{cases}.$$

This is a square well with a bottom that slopes downwards from left to right with a slope of $-\alpha$, and a top beginning at $x = a$ that slopes down from $V_0$ with a slope of $-\alpha$. A particle of energy $E$ could then tunnel out after the point $x_0 = (V_0 - E)/\alpha$.

c) The limits of integration are from $a$ to $x_0$, and $p(x) = \sqrt{2m(V_0 - E - \alpha x)}$. Thus,

$$\gamma = \frac{2\sqrt{2m}}{3\hbar\alpha}(V_0 - \alpha a - E)^{3/2}.$$

Since $V_0 \gg \alpha a + E$, we get that

$$\gamma \approx \frac{2\sqrt{2m}}{e\hbar\alpha} V_0^{3/2}.$$

To compute $\tau$, assume that all energy is kinetic, so that $v = \pi\hbar/(2ma)$, using the ground state energy above. We also have that $x_1$ is just $x_0 - a$, so that $\tau \sim 10^{49}$ s, which is 32 orders of magnitude greater than the age of the universe.

**Exercise 5. Semiclassical Approximations.** Consider a single particle moving in one dimension. In this problem, you will analyze the semiclassical behavior of such a particle.

a) Make the ansatz

$$\psi = e^{\frac{i}{\hbar}\sigma},$$

and obtain a differential equation for $\sigma$ from the time-independent Schrödinger equation.

b) Look for solutions to the equation from (a) of the form

$$\sigma = \sigma_0 + \frac{\hbar}{i}\sigma_1 + \left(\frac{\hbar}{i}\right)^2 \sigma_2 + \cdots .$$

Proceeding as in perturbation theory, show that the zeroth order in $\hbar$ equation is

$$\frac{1}{2m}\sigma_0'^2 = E - V.$$

c) Solve the equation for $\sigma_0$, and give the condition of validity of your solution. Show that this condition can be written

$$\frac{d}{dx}\left(\frac{\lambda}{2\pi}\right) \ll 1,$$

where $\lambda$ is the de Broglie wavelength of the particle.

d) By writing $dp/dx$ in terms of the classical force $F$, show that the condition from (c) can be written

$$\frac{m\hbar F}{p^3} \ll 1.$$

Argue that this implies that the semiclassical approximation is not valid near turning points of the potential, where we have to solve Airy's equation and use the connecting solutions.

e) Obtain the first order equation, and show that it has the solution

$$\sigma_1 = -\frac{1}{2}\ln(p).$$

f) Thus, show that, to first order in $\hbar$, you recover the WKB approximation

$$\psi = \frac{C_1}{\sqrt{p}} e^{\frac{i}{\hbar}\int p dx} + \frac{C_2}{\sqrt{p}} e^{-\frac{i}{\hbar}\int p dx}.$$

g) Give a physical interpretation of the $\frac{1}{\sqrt{p}}$ factor which appears in $\psi$.

a) We plug in to the Schrödinger equation to find

$$\sigma'^2 + \frac{\hbar}{i}\sigma'' = 2m(E - V).$$

b) Just plug in the power series for $\sigma$. Note that the $\sigma''$ term is of first order in $\hbar/i$ so doesn't contribute to the zeroth order equation.

c) We have

$$\sigma_0' = \pm p.$$

Thus,

$$\sigma_0 = \pm \int p dx + C.$$

We'll drop the integration constant in the sequel (i.e. assume some limits of integration). In order to ignore the $\sigma''$ term from the equation in part (a), we need

$$\hbar \frac{\sigma''}{\sigma'^2} \ll 1.$$

Thus,

$$\frac{d}{dx}\left(\frac{\hbar}{\sigma'}\right) = \frac{d}{dx}\left(\frac{\hbar}{p}\right) \ll 1.$$

Since $p = 2\pi\hbar/\lambda$, we have the result.

d) We write

$$\frac{dp}{dx} = \frac{d}{dx}\sqrt{2m(E - V)} = -\frac{m}{p}\frac{dV}{dx} = \frac{mF}{p}.$$

Thus,

$$\frac{m\hbar F}{p^3} \ll 1$$

as desired. Now, near the turning points, $p$ is small while $F$ is a constant. Indeed, near a turning point $V$ is approximately linear, so $F$ is some nonzero constant. On the other hand, near the turning point the classical velocity (hence momentum) is small. Thus, the expression

$$\frac{m\hbar F}{p^3}$$

will be large near turning points, so we can't use the semiclassical approximation.

e) The first order equation is

$$\sigma_0'\sigma_1' + \frac{\sigma_0''}{2} = 0.$$

Since $\sigma_0' = p$, we have

$$\sigma_1' = -\frac{p'}{2p}.$$

This has the solution

$$\sigma_1 = -\frac{1}{2}\ln(p),$$

which can be obtained directly by integration.

f) Since to first order $\psi = e^{\frac{i}{\hbar}\sigma_0 + \frac{\hbar}{i}\sigma_1}$, we need only plug in the expressions $\sigma_0 = \pm\int p\,dx$ and $\sigma_1 = \ln(1/\sqrt{p})$. Note that the two linearly independent solutions for $\sigma_0$ give the two different signs in the exponentials in $\psi$.

g) The probability of finding the particle between $x$ and $x + dx$ is $|\psi|^2 \sim \frac{1}{p}$. Classically, the amount of time a particle spends in the interval $dx$ is inversely proportional to the velocity (or the momentum) of the particle.

**Exercise 6. Hard Sphere Scattering Continued.** In Example 10.3, Griffiths computes for hard sphere scattering that if $ka \ll 1$, where $a$ is the radius of the sphere, then

$$\sigma \approx 4\pi a^2.$$

Show that in the other limit, $ka \gg 1$,

$$\sigma \approx 2\pi a^2.$$

Note that

$$\sigma = \frac{4\pi}{k^2}\sum_{\ell=0}^{\ell_{\max}}(2\ell+1)\sin^2\delta_\ell,$$

where $\ell_{\max}$ is the maximum value of $\ell$ which for large $ka$ we can take to be $\ell_{\max} = ka$. Now, Griffiths computes that

$$\delta_\ell = \arctan\left[\frac{j_\ell(ka)}{n_\ell(ka)}\right].$$

As $ka \to \infty$,

$$\begin{aligned} j_\ell \to& \frac{1}{ka}\sin\left(ka - \frac{\ell\pi}{2}\right) \\ n_\ell \to& -\frac{1}{ka}\cos\left(ka - \frac{\ell\pi}{2}\right). \end{aligned}$$

Thus,

$$\delta_\ell \to \frac{\ell\pi}{2} - ka,$$

and

$$\sin^2\delta_\ell \to \sin^2(ka - \ell\pi/2).$$

Approximating the sum over $\ell$ in $\sigma$ by an integral and $2\ell + 1 \approx 2\ell$, we get

$$\begin{aligned} \sigma \approx& \frac{4\pi}{k^2}\int_0^{ka} 2\ell\sin^2\delta_\ell d\ell \\ =& \frac{4\pi}{k^2}\int [\ell - \ell\cos(\ell\pi - 2ka)]d\ell \\ =& \frac{4\pi}{k^2}\left[\frac{(ka)^2}{2} - \frac{\ell}{\pi}\sin(\ell\pi - 2ka)\Big|_0^{ka} + \frac{1}{\pi}\int_0^{ka}\sin(\ell\pi - 2ka)d\ell\right] \\ \approx& 2\pi a^2, \end{aligned}$$

where in the last approximation we drop all terms of order $< 2$ in $ka$ (i.e. all terms but the first term in the second-to-last line).

**Exercise 7. Soft Sphere Scattering.**

a) Consider scattering from a finite spherical well of depth $V_0$ and radius $a$. Show that the s-wave phase shift is

$$\delta_0 = -ka + \arctan\left[\frac{k}{k'}\tan(k'a)\right],$$

where $k$ and $k'$ are the wave numbers outside and inside the well, respectively.

b) Suppose that $k$ is small so that we can ignore $ka$, and consider varying the depth of the well, i.e. varying $k'$. Show that whenever

$$k' \approx {k'}_n \equiv \frac{(2n+1)\pi}{2a},$$

the phase shift becomes

$$\delta_0 = \delta_b + \arctan\left(\frac{\Gamma/2}{E_0 - E}\right), \tag{1}$$

where we can ignore $\delta_b$, the **background phase**, and $\Gamma/2 = \hbar^2 k_n/ma$ ($k_n$ is the value of $k$ when $k' = k'_n$). You should give a reasonable definition of $E_0$.

c) Assuming the form for $\delta_0$ in Equation 1 and ignoring $\delta_b$, show that

$$\sigma = \frac{4\pi}{k^2}\frac{(\Gamma/2)^2}{(E_0 - E)^2 + (\Gamma/2)^2},$$

so that $\Gamma$ is the width of the bell curve which describes $\sigma$. This is called the **Breit-Wigner form** for the cross-section and describes the phenomenon of **resonance**.

d) If we start with $V_0$ too small to support a bound state, show that $k'_1$ corresponds to the well developing its first bound state (at zero energy, i.e. at $k = 0$). As the well is deepened further, another zero energy bound state is formed at $k'_2$.
*Hint*: It may be helpful at this point to recall your solution for (or solve, if you haven't before) Exercise 4.11 in Griffiths.

a) The radial equation inside the well is

$$u'' = -(E + V_0)\frac{2m}{\hbar^2}u,$$

so that the s-state radial wavefunction inside the well is

$$\psi_{\text{in}}(r) = Aj_0(k'r),$$

where

$$k' = \sqrt{2m(E + V_0)}/\hbar.$$

Note that there is no $n_0$ term since $n_0$ blows up at the origin. Outside the well, the radial wavefunction is just given by a linear combination of spherical Bessel functions:

$$\psi_{\text{out}}(r) = Bj_0(kr) + Cn_0(kr),$$

where

$$k = \sqrt{2mE}/\hbar.$$

Since the potential is finite, we must have continuity of the wavefunction and its first derivative at $r = a$. Continuity at $a$ gives

$$A = \frac{k'}{\sin(k'a)}\left[B\frac{\sin(ka)}{k} - C\frac{\cos(ka)}{k}\right].$$

Plugging this in for $A$ in the equation for continuity of the first derivative, we find

$$\frac{k'}{k}\left[B\sin(ka) - C\cos(ka)\right]\left[\cot(k'a) - \frac{1}{k'a}\right] = B\left[\cos(ka) - \frac{\sin(ka)}{ka}\right] + C\left[\sin(ka) + \frac{\cos(ka)}{ka}\right].$$

Now,

$$\delta_0 = \arctan(-C/B).$$

Thus, we can solve the equation above for $-C/B$ to find

$$-\frac{C}{B} = \frac{-\tan(ka) + \frac{k}{k'}\tan(k'a)}{1 + \frac{k}{k'}\tan(ka)\tan(k'a)}.$$

Next, you have to recall (or look up) the following identity for arctan.

$$\arctan(x) + \arctan(y) = \arctan\left(\frac{x+y}{1-xy}\right).$$

Using this identity, we have the desired result.

b) Since $k'$ is near $k'_n$, we know that $k'_n - k' = \varepsilon$ is a small quantity. Recall that

$$E = \frac{\hbar^2 k'^2}{2m} - V_0,$$

and define

$$E_0 = \frac{\hbar^2 {k'_n}^2}{2m} - V_0$$

to be the energy associated to $k'_n$. Then

$$E_0 - E = \frac{\hbar^2}{m}k'_n\varepsilon,$$

or

$$\varepsilon = \frac{m}{\hbar^2 k'_n}(E_0 - E).$$

Further,

$$\tan(k'a) \approx \frac{1}{a\varepsilon}.$$

Plugging this in to the form for $\delta_0$ obtained and ignoring factors of $\varepsilon^2$, we find

$$\frac{k}{k'_n - \varepsilon}\cdot\frac{1}{a\varepsilon} = \frac{\hbar^2 k_n}{ma(E_0 - E)},$$

where we make the approximation $k \approx k_n$.

c) Recall that

$$\sigma = \frac{4\pi}{k^2} \sin^2 \delta_0.$$

Since

$$\sin(\arctan(x)) = \frac{x}{\sqrt{1+x^2}},$$

we have

$$\sigma = \frac{4\pi}{k^2} \frac{1}{\frac{(E_0-E)^2}{(\Gamma/2)^2} + 1},$$

which is the desired form after multiplying through by $(\Gamma/2)^2$.

d) In Exercise 4.11, it was found that the finite spherical well cannot support a bound state if $V_0 < \pi\hbar^2/8ma^2$. There was a typo, that we actually want to consider $n = 0$ instead of $n = 1$ for the first bound state. Indeed, if $E = 0$, then

$$0 = \frac{\hbar^2 k_n'^2}{2m} - V_0,$$

so

$$V_0 = \frac{\hbar^2 \pi}{8ma^2}$$

is the minimum energy for the first bound state, which occurs at $k'_0$, not $k'_1$.