Week 12 Worksheet Solutions

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Exercise 1. The integral form of the Schrödinger equation reads

$$\psi(\mathbf{r}) = \psi_0(\mathbf{r}) + \int g(\mathbf{r} - \mathbf{r}')V(\mathbf{r}')\psi(\mathbf{r}')\,\mathrm{d}^3\mathbf{r}',$$

where

$$g(\mathbf{r}) = -\frac{m}{2\pi\hbar^2} \cdot \frac{e^{ikr}}{r}$$

is the Green's function for the Schrödinger equation.

a) Use the method of successive approximations to write $\psi(\mathbf{r})$ as a series in the incident wavefunction $\psi_0(\mathbf{r})$.

b) Truncate the Born series you obtain after the second term to get the first Born approximation. Assuming the potential is localized near $\mathbf{r}' = 0$, we can write

$$\frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r} - \mathbf{r}'|} \approx \frac{e^{ikr}}{r} e^{-i\mathbf{k}\cdot\mathbf{r}'}.$$

Using this and the definition of $f(\theta)$,

$$\psi(\mathbf{r}) = Ae^{ikz} + f(\theta)\frac{e^{ikr}}{r},$$

determine $f(\theta)$.

c) In Griffiths, we find that for a potential $V(r) = V_0/r$, $f_{\text{point}}(\theta) = -\frac{2mV_0}{\hbar^2 q^2}$, where $\mathbf{q} = \mathbf{k}' - \mathbf{k}$. If $V(\mathbf{r}) = -e^2 Z/r$ for an electron scattering off a point charge of charge Ze, how would $f(\theta)$ change if instead the electron scatters off a spherical nucleus of radius a, charge Ze, and uniform charge density? Your answer should be of the form

$$f(\theta) = f_{\text{point}}(\theta) \cdot F(q),$$

where $F(q)$ is the **form factor** of the nucleus.

d) If you haven't done so already, calculate $F(q)$ explicitly.

e) From scattering high-energy electrons at nuclei, the actual form factor is measured to be

$$F(q) = \frac{Ze}{(1+q^2{a_N}^2)^2},$$

where $a_N \approx 0.26$ fm. If the inverse Fourier transform of $\frac{1}{(1+x^2)^2}$ is $e^{-|x|}$, what does that tell you about the size and charge density of the proton?

a) The idea is to plug in the formula for ψ for the $\psi(\mathbf{r}')$ on the right side. Thus, we obtain

$$\begin{aligned}\psi(\mathbf{r}) =& \psi_0(\mathbf{r}) + \int g(\mathbf{r}-\mathbf{r}')V(\mathbf{r}')\psi_0(\mathbf{r}')\,\mathrm{d}^3r' + \\ &+ \iint g(\mathbf{r}-\mathbf{r}')g(\mathbf{r}'-\mathbf{r}'')V(\mathbf{r}')V(\mathbf{r}'')\psi_0(\mathbf{r}'')\,\mathrm{d}^3r'\,\mathrm{d}^3r'' + \cdots .\end{aligned}$$

b) The Born approximation is then

$$\psi(\mathbf{r}) \approx \psi_0(\mathbf{r}) + \int g(\mathbf{r}-\mathbf{r}')V(\mathbf{r}')\psi_0(\mathbf{r}')\,\mathrm{d}^3r'.$$

Plugging in the suggested approximation for the Green's function, we get

$$\psi_0(\mathbf{r}) - \frac{m}{2\pi\hbar^2}\int \frac{e^{ikr}}{r}e^{-i\mathbf{k}\cdot\mathbf{r}'}V(\mathbf{r}')\psi_0(\mathbf{r}')\,\mathrm{d}^3r'.$$

Now, suppose the incident wavefunction is a plane wave e^{ikz} along the $\hat{z}$ direction. It follows then that

$$\psi(\mathbf{r}) \approx e^{ikz} - \frac{m}{2\pi\hbar^2}\int \frac{e^{ikr}}{r}e^{-i\mathbf{k}\cdot\mathbf{r}'}V(\mathbf{r}')e^{i\mathbf{k}'\cdot\mathbf{r}'}\,\mathrm{d}^3r',$$

where we set $\mathbf{k}' = k\hat{z}$. Thus,

$$\psi(\mathbf{r}) = e^{ikz} + \frac{e^{ikr}}{r}\cdot\left(-\frac{m}{2\pi\hbar^2}\int e^{i(\mathbf{k}'-\mathbf{k})\cdot\mathbf{r}'}V(\mathbf{r}')\,\mathrm{d}^3r'\right),$$

so

$$f(\theta) = -\frac{m}{2\pi\hbar^2}\int e^{i(\mathbf{k}'-\mathbf{k})\cdot\mathbf{r}'}V(\mathbf{r}')\,\mathrm{d}^3r'.$$

c) Uniform charge density implies $\rho(r) = \frac{Ze}{\frac{4}{3}\pi a^3}$ for $r \in (0, a)$. Thus,

$$V(\mathbf{r}) = -e\int \frac{1}{|\mathbf{r}-\mathbf{r}''|}\rho(\mathbf{r}'')\,\mathrm{d}^3r''.$$

Plugging this in to the expression for $f(\theta)$ from part (b), we get

$$\begin{aligned}f(\theta) =& -\frac{m}{2\pi\hbar^2}\int e^{i\mathbf{q}\cdot\mathbf{r}'}V(\mathbf{r}')\,\mathrm{d}^3r' \\ =& \frac{me}{2\pi\hbar^2}\iint e^{i\mathbf{q}\cdot\mathbf{r}'}\frac{\rho(\mathbf{r}'')}{|\mathbf{r}'-\mathbf{r}''|}\,\mathrm{d}^3r'\,\mathrm{d}^3r''.\end{aligned}$$

Make the substitution $\mathbf{u} = \mathbf{r}' - \mathbf{r}''$, so

$$\begin{aligned} f(\theta) =& \frac{me}{2\pi\hbar^2} \iint e^{i\mathbf{q}\cdot\mathbf{u}} e^{i\mathbf{q}\cdot\mathbf{r}''} \frac{\rho(\mathbf{r}'')}{|\mathbf{u}|} \, \mathrm{d}^3u \, \mathrm{d}^3r'' \\ =& f_{\text{point}}(\theta) \frac{1}{Ze} \int e^{i\mathbf{q}\cdot\mathbf{r}''} \rho(\mathbf{r}'') \, \mathrm{d}^3r'' \implies \\ \implies F(q) =& \frac{1}{eZ} \int e^{i\mathbf{q}\cdot\mathbf{r}''} \rho(\mathbf{r}'') \, \mathrm{d}^3r''. \end{aligned}$$

Note that you can avoid doing the integral in the second equality since the formula for $f_{\text{point}}(\theta)$ is given in the problem statement. goes as follows. We have

$$\int \frac{e^{i\mathbf{q}\cdot\mathbf{u}}}{u} \, \mathrm{d}^3u = \int e^{iqu\cos\theta} u \cdot 2\pi \, \mathrm{d}(\cos\theta) \, \mathrm{d}u,$$

where we can assume that the angle between $\mathbf{q}$ and $\mathbf{u}$ is θ since we're integrating over all θ anyway. Now, do the θ integral to get

$$\int \frac{2\pi}{iq} (e^{iqu} - e^{-iqu}) \, \mathrm{d}u = \frac{4\pi}{q} \int_0^\infty \sin(qu) \, \mathrm{d}u = \frac{4\pi}{q^2},$$

where the final equality follows by regulating the undefined integral with an exponential e^{-au} and then setting $a \to 0$ in the answer.

d) We plug in the form for ρ above and calculate.

$$\begin{aligned} F(q) =& \frac{3}{4\pi a^3} \int_0^{2\pi} \mathrm{d}\varphi \int_0^\pi \mathrm{d}\theta \int_0^a r^2 \sin\theta e^{iqr\cos\theta} \, \mathrm{d}r \\ =& \frac{3}{2a^3} \int_0^a r^2 \, \mathrm{d}r \frac{1}{iqr} (e^{iqr} - e^{-iqr}) \\ =& \frac{3}{2a^3} \int_0^a \mathrm{d}r \frac{2r}{q} \sin(qr) \\ =& \frac{3}{a^3q^3} \left(\sin(qa) - aq\cos(qa) \right). \end{aligned}$$

e) Note that

$$f(\theta) = f_{\text{point}}(\theta) \frac{1}{eZ} \int e^{i\mathbf{q}\cdot\mathbf{x}} \rho(\mathbf{x}) d^3x.$$

Thus, $F(q)$ is $\frac{1}{eZ}$ times the Fourier transform of ρ. If it's given by

$$\frac{Ze}{(1 + q^2 {a_N}^2)^2},$$

then

$$\rho(\mathbf{r}) = Zee^{-r/a_N}.$$

Thus, we find that the charge density of the nucleus has an *exponential* distribution! The proton is then "smeared out" over all space, but it has a $1/e$ drop off after $r = a_N$. So, we can consider the "size" of the proton to be $\approx a_N$.