Week 12 Worksheet
(More) Born Approximation

Jacob Erlikhman

November 12, 2025

Exercise 1. The integral form of the Schrödinger equation reads

$$\psi(\mathbf{x}) = \psi_0(\mathbf{x}) + \int G_0(\mathbf{x}, \mathbf{x}', E) V(\mathbf{x}') \psi(\mathbf{x}')\, \mathrm{d}^3 x',$$

where

$$G_0(\mathbf{x}, \mathbf{x}', E) = -\frac{m}{2\pi\hbar^2} \cdot \frac{e^{ik|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|}$$

is the Green's function for the free-particle Schrödinger equation.

a) Use the method of successive approximations to write $\psi(\mathbf{x})$ as a series in the incident wavefunction $\psi_0(\mathbf{x})$.

b) Truncate the Born series you obtain after the second term to get the first Born approximation. Assuming the potential is localized near $\mathbf{r}' = \mathbf{0}$, we can write

$$\frac{e^{ik|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|} \approx \frac{e^{ikr}}{r} e^{-i\mathbf{k}\cdot\mathbf{x}'},$$

where $r = |\mathbf{x}|$. Using this and the definition of $f(\theta)$,

$$\psi(\mathbf{r}) = Ae^{ikz} + f(\theta)\frac{e^{ikr}}{r},$$

determine $f(\theta)$.

c) In Griffiths, we find that for a potential $V(r) = V_0/r$, $f_{\text{point}}(\theta) = -\frac{2mV_0}{\hbar^2 q^2}$, where $\mathbf{q} = \mathbf{k}' - \mathbf{k}$. If $V(\mathbf{r}) = -e^2 Z/r$ for an electron scattering off a point charge of charge Ze, how would $f(\theta)$ change if instead the electron scatters off a spherical nucleus of radius a, charge Ze, and (not necessarily uniform) charge density $\rho(\mathbf{x})$? Write your answer in the form

$$f(\theta) = f_{\text{point}}(\theta) \cdot F(q),$$

where $F(q)$ is the **form factor** of the nucleus.
Hint: To make sense of an integral

$$\int_0^\infty \sin(x) dx,$$

use a **regulator**: Instead, calculate

$$\int_0^\infty \sin(x) e^{-\alpha x} dx,$$

and set $\alpha \to 0$ in your final answer.

d) In the case that ρ is uniform, calculate $F(q)$ explicitly.

e) From scattering high-energy electrons at nuclei, the actual form factor is measured to be

$$F(q) = \frac{1}{(1 + q^2 {a_N}^2)^2},$$

where $a_N \approx 0.26$ fm. If the inverse Fourier transform of $\frac{1}{(1+k^2)^2}$ is $e^{-|x|}$, what does that tell you about the size and charge density of the proton?