Week 2 Worksheet Solutions
Identical Particles

Jacob Erlikhman

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Exercise 1.

Write down the hamiltonian for two noninteracting identical particles in the infinite square well. Write down the ground states for the three cases: distinguishable, fermions, bosons. Recall that the one-particle wavefunctions are

$$\psi_n(x) = \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi}{a} x\right),$$

with energies $E_n = n^2\pi^2\hbar^2/2ma^2$.

Find the first three excited states and their energies for each of the three cases (distinguishable, fermions, bosons).

a) $H = \frac{{p_1}^2}{2m} + \frac{{p_2}^2}{2m} + V(x_1, x_2)$, where

$$V(x_1, x_2) = \begin{cases} 0, & x_1, x_2 \in [0, a] \\ \infty, & x_1 > a \text{ or } x_2 > a \end{cases}.$$

Distinguishable is just $\psi_1(x_1)\psi_1(x_2)$ (ground), $\psi_1(x_1)\psi_2(x_2)$ (1st and 2nd excited, along with the same with $x_1 \leftrightarrow x_2$), and $\psi_2(x_1)\psi_2(x_2)$ (3rd excited). Their energies are $2E_1$, $5E_1$, $5E_1$, and $8E_1$, respectively.

b) For bosons, we get almost the same thing. The 1st and 2nd excited states now merge into $(\psi_1(x_1)\psi_2(x_2) + \psi_2(x_1)\psi_1(x_2))/\sqrt{2}$. Thus, the ground state is the same, the 1st excited state is the one in the previous sentence, the second excited state is $\psi_2(x_1)\psi_2(x_2)$, and the third is $(\psi_3(x_1)\psi_1(x_2) + \psi_1(x_1)\psi_3(x_2))/\sqrt{2}$. Their energies are $2E_1$, $5E_1$, $8E_1$, and $10E_1$, respectively.

c) For fermions, any manifestly symmetric state is now not allowed. Thus, the ground state is $(\psi_1(x_1)\psi_2(x_2) - \psi_2(x_1)\psi_1(x_2))/\sqrt{2}$, the first excited is $(\psi_1(x_1)\psi_3(x_2) - \psi_3(x_1)\psi_1(x_2))/\sqrt{2}$, the second excited is $(\psi_2(x_1)\psi_3(x_2) - \psi_3(x_1)\psi_2(x_2))/\sqrt{2}$, and the third excited is $(\psi_1(x_1)\psi_4(x_2) - \psi_4(x_1)\psi_1(x_2))/\sqrt{2}$. Their energies are $5E_1$, $10E_1$, $13E_1$, and $17E_1$, respectively.

Exercise 2. In Exercise 2, we ignored spin (or at least supposed that the particles are in the same spin state).

a) Do it now for particles of spin 1/2. Construct the four lowest-energy configurations, and specify their energies and degeneracies.

b) Do the same for spin 1.

a) If you remember (or look up) the triplet and singlet states, you'll find that there are 3 symmetric ones and 1 antisymmetric one. Hence, we can pair a symmetric wavefunctiton with the antisymmetric spin state to get an antisymmetric state; conversely, we can pair an antisymmetric wavefunction with a symmetric spin state (in 3 different ways) to also get an antisymmetric state. Thus, the ground state for spin 1/2 particles is $\psi_1(x_1)\psi_1(x_2)$ paired with the singlet. It has multiplicity 1 and energy $2E_1$. The next highest state is $(\psi_1(x_1)\psi_2(x_2)-\psi_2(x_1)\psi_1(x_2))/\sqrt{2}$ paired with the triplet *or* $(\psi_1(x_1)\psi_2(x_2)+\psi_2(x_1)\psi_1(x_2))/\sqrt{2}$ paired with the singlet. Hence, these states all have energy $5E_1$ and multiplicity 4. Continuing with the game, the next highest energy state is $\psi_2(x_1)\psi_2(x_2)$ paired with the singlet. This has multiplicity 1 and energy $8E_1$. Finally, the 3rd highest energy states are $(\psi_1(x_1)\psi_3(x_2)-\psi_3(x_1)\psi_1(x_2))/\sqrt{2}$ paired with the triplet or $(\psi_1(x_1)\psi_3(x_2)+\psi_3(x_1)\psi_1(x_2))/\sqrt{2}$ paired with the singlet, which again have multiplicity 4 and energy $10E_1$.

b) We can determine how many symmetric/antisymmetric combinations we have by considering the possible spin values. Each particle can have spin -1, 0, or 1. Thus, there are 6 symmetric spin states and 3 antisymmetric, where the symmetric ones correspond to either taking pairs with the same spin or the symmetric combination of a pair with different spins (e.g. $\frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$). Other than this, the game is the same as in part (a). We get the same wavefunctions, but the multiplicities are instead 6, 9, 6, 9.