Week 3 Worksheet Solutions
Identical Particles Continued

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Exercise 1. Symmetries of Many-Particle States.

a) Consider a system of two identical particles. Define a **permutation operator** via

$$P_{12} |\alpha\rangle |\beta\rangle = |\beta\rangle |\alpha\rangle .$$

Show that ${P_{12}}^2 = \mathbb{1}$, the identity operator, and that the eigenvalues of P_{12} are ± 1. Thus, show that its eigenvectors are either totally symmetric or antisymmetric.

b) Generalize part (a) to systems of three identical particles. You should find that you have *six* permutation operators. Assuming the hamiltonian is invariant under each of these operators, is there a complete set of common eigenvectors?

c) ***Griffiths 5.8.*** In the situation of (b), suppose that the particles have access to three distinct one-particle states, $|a\rangle$, $|b\rangle$, and $|c\rangle$. For example, $|abc\rangle$ is an allowed state, as is $|aaa\rangle$. How many states can be constructed if they are (i) bosons or (ii) fermions?

d) Suppose we have a single-particle fermion state $|\alpha\rangle$ and a single-particle bosonic state $|\beta\rangle$. Just like for the harmonic oscillator, we can define **creation operators** ${C_\alpha}^\dagger$ and ${a_\beta}^\dagger$, such that given any state $|\psi\rangle$,

$$\begin{aligned} {C_\alpha}^\dagger |\psi\rangle &= |\alpha\psi\rangle \\ {a_\beta}^\dagger |\psi\rangle &= |\beta\psi\rangle . \end{aligned}$$

The operators ${C_\alpha}^\dagger$ and ${a_\beta}^\dagger$ have the following properties.

$$\begin{aligned} C_\alpha |\alpha\psi\rangle &= |\psi\rangle \\ a_\beta |\beta\psi\rangle &= |\psi\rangle \\ C_\alpha |0\rangle = a_\beta |0\rangle &= 0 \\ {C_\alpha}^\dagger {C_\alpha}^\dagger &= 0 \\ \{C_\alpha, {C_{\alpha'}}^\dagger\} \equiv C_\alpha {C_{\alpha'}}^\dagger + {C_{\alpha'}}^\dagger C_\alpha &= \delta_{\alpha\alpha'} \mathbb{1} \\ \{{C_\alpha}^\dagger, {C_{\alpha'}}^\dagger\} &= 0 \\ [a_\beta, {a_{\beta'}}^\dagger] &= \delta_{\alpha\alpha'} \mathbb{1} \\ [{a_\beta}^\dagger, {a_{\beta'}}^\dagger] &= 0, \end{aligned}$$

where $|0\rangle$ denotes a state with no particles at all. To what extent is a bound pair of fermions equivalent to a boson?
Hint: Use the symmetries of many-particle states and the (anti-)commutation relations of the creation/annihilation operators constructed in parts (a)-(d). What algebra must the creation/annihilation operators for the bound pair satisfy?

e) Prove the properties given in (d).
Hints: It may be useful to use the notation $\sim \alpha$ for the α "orbital" being *unoccupied*. To show the first relation for C_α, try to first show that $C_\alpha |\alpha\rangle = |0\rangle$. For the anti-commutator relations, consider separately the cases $\alpha \neq \alpha'$ and whether the α or α' orbitals are occupied.

a) ${P_{12}}^2 = \mathbb{1}$ follows by just applying it twice to the state. Suppose $|\lambda\rangle$ is an eigenvector of P_{12} with eigenvalue $\lambda \in \mathbb{C}$. Then

$$\lambda^2 = 1,$$

so $\lambda \in \{-1, 1\}$. An eigenvector with eigenvalue 1 will be symmetric, while an eigenvector with eigenvalue -1 will be antisymmetric.

b) We want to consider exchanges of three particles, so we will have $3! = 6$ distinct permutations. There is the identity operator, pairwise interchange, P_{12}, P_{23}, P_{13}, and the two cyclic permutations P_{123}, ${P_{123}}^2$. Since the permutation operators are not mutually commuting, we don't have a complete set of common eigenvectors. Instead, the space divides into four **invariant subspaces**, which have the property that any vector in an invariant subspace is transformed by the operators into a vector which is *in the same subspace*. Two of the subspaces are partially symmetric (and hence can be ignored), while the other two are the symmetric subspace and the antisymmetric one.

c) For bosons we have $10 = 6 + 3 + 1$ states. 6 states are symmetrizations of states of type (x, x, y), with $x, y \in \{a, b, c\}$ and $x \neq y$. We get $3 \cdot 2$ states of this type. The other type is (a, b, c), and there is only one state of this kind since we need to include all possible permutations to symmetrize. Finally, there are 3 states of the form (x, x, x).

For fermions, only 1 state is possible, since any state with repeating letters cannot be anti-symmetrized. Hence, the only type of allowed state is the anti-symmetrization of (a, b, c).

d) First some comments. In the Standard Model, quarks (which are fermions) can bind together to form particles. Some of these obey fermionic statistics, like the proton, while some obey bosonic statistics, like the pion. So two fermions can bind together to form a boson in principle, and this is really what happens in our world! We will see in this problem that the creation/annihilation operator pair for a bound pair of fermions (e.g. a pair of quarks which bind to make a boson) does not quite obey a bosonic operator algebra!

If we consider only the single bound state by itself, i.e. the one-particle state, then there is no difference. Indeed, the spin is an integer, and there is no other requirement for bosons until we consider multi-particle states. Thus, consider a two-particle system made up of two such bound pairs. Under interchange of the pairs, we are effectively interchanging two fermions twice; thus, the composite system will be multiplied by $(-1)^2 = 1$. It follows that under interchange of pairs, the system obeys bosonic statistics. This clearly generalizes to systems of many pairs. Consider now the commutation relations between the creation and annihilation operators which make such a state. We need to consider the algebra generated by $D_{\alpha\alpha'} = C_\alpha C'_\alpha$ and $D_{\alpha\alpha'}{}^+ = C_\alpha{}^\dagger C_{\alpha'}{}^\dagger$. Notice that $D_{\alpha\alpha'}{}^\dagger \neq D_{\alpha\alpha'}{}^+$! First of all, note that $(D^\dagger)^2 = D^2 = 0$, which is distinct from the bosonic creation operators. What about (anti-)commutators? Consider

$$[D_{12}, D_{12}{}^\dagger].$$

This is given by

$$\begin{aligned}[C_1C_2, C_2{}^\dagger C_1{}^\dagger] =& C_1C_2C_2{}^\dagger C_1{}^\dagger - C_2{}^\dagger C_1{}^\dagger C_1 C_2 = \\ =& - C_1C_2{}^\dagger C_2 C_1{}^\dagger + C_1C_1{}^\dagger + C_2{}^\dagger C_1 C_1{}^\dagger C_2 - C_2{}^\dagger C_2 = C_1C_1^\dagger - C_2^\dagger C_2 = \\ =& \mathbb{1} - C_1^\dagger C_1 - C_2^\dagger C_2.\end{aligned}$$

Now, since by Pauli exclusion we can't have more than one fermion in state 1 or 2, we should consider how $A = [D_{12}, D_{12}{}^\dagger]$ acts on states of the three kinds: no fermions in states 1 or 2, one of the states filled, both states filled. If there is no particle in states 1 or 2, then A acts by the identity. On the other hand, if there is a particle in one of these states, then A acts by 0 (resp. by $-\mathbb{1}$) if one of the states is filled (resp. both states are filled). Moreover, we expect that $[a, a^\dagger] = \mathbb{1}$ for a bosonic creation-annihilation operator pair $a^\dagger, a$, so these $D_{\alpha\alpha'}$ operators are not quite bosonic creation operators. For more on this, see 🔗 **this article** and the references therein.

e) $C_\alpha |\alpha\psi\rangle = |\psi\rangle$: Consider the state $C_\alpha |\alpha\rangle = (\langle\alpha| C_\alpha{}^\dagger)^\dagger$. Note that the state $\langle\alpha| C_\alpha{}^\dagger$ satisfies

$$\langle\alpha| C_\alpha{}^\dagger |\psi\rangle = \delta_{0,\psi}.$$

Hence, $(\langle\alpha| C_\alpha{}^\dagger)^\dagger = C_\alpha |\alpha\rangle = |0\rangle$. If instead of $|\alpha\rangle$ we use $|\alpha\psi\rangle$, we end up with the desired relation.

$a_\beta |\beta\psi\rangle = |\psi\rangle$: The proof of this relation is identical to the previous one.

$C_\alpha |0\rangle = a_\beta |0\rangle = 0$: We have that

$$\langle 0| C_\alpha{}^\dagger |\psi\rangle = 0$$

for any ψ. Hence,

$$\langle 0| C_\alpha{}^\dagger = 0 \implies C_\alpha |0\rangle = (\langle 0| C_\alpha{}^\dagger)^\dagger = 0.$$

$C_\alpha{}^\dagger C_\alpha{}^\dagger = 0$: The property $(C_\alpha{}^\dagger)^2 = 0$ follows from the fact that fermions are antisymmetric under interchange.

$\{C_\alpha, C_{\alpha'}{}^\dagger\} = \delta_{\alpha\alpha'}\mathbb{1}$: Consider first the case $\alpha \neq \alpha'$. Then it's clear we get 0 if either the α orbital is empty or the α' orbital is occupied. So it's sufficient to consider its effect on a vector $|\alpha \cdots \sim \alpha'\rangle$. This gives

$$\begin{aligned}\{C_\alpha{}^\dagger, C_\alpha\} |\alpha \cdots \sim \alpha'\rangle =& C_\alpha |\alpha'\alpha \cdots\rangle + C_{\alpha'}{}^\dagger |\cdots \sim \alpha \sim \alpha'\rangle \\ =& - C_\alpha |\alpha\alpha' \cdots\rangle + C_{\alpha'}{}^\dagger |\cdots \sim \alpha \sim \alpha'\rangle \\ =& - |\alpha' \cdots \sim \alpha\rangle + |\alpha' \cdots \sim \alpha\rangle = 0.\end{aligned}$$

For the case $\alpha = \alpha'$, consider separately the cases of the α orbital being occupied or empty. We have

$$\{C_\alpha{}^\dagger, C_\alpha\} |\alpha \cdots\rangle = 0 + C_\alpha{}^\dagger |\cdots \sim \alpha\rangle = |\alpha \cdots\rangle$$
$$\{C_\alpha{}^\dagger, C_\alpha\} |\cdots \sim \alpha\rangle = C_\alpha |\alpha \cdots\rangle + 0 = |\cdots \sim \alpha\rangle .$$

Thus, $\{C_\alpha{}^\dagger, C_\alpha\}$ is the identity operator.

$[a_\beta, a_{\beta'}{}^\dagger] = \delta_{\alpha\alpha'}\mathbb{1}$: The key to figuring out this relation is to show that the creation operators and annihilation operators are the same as those for the harmonic oscillator. For example, consider $a_\beta{}^\dagger$. If it acts on $|0\rangle$, we get $|\beta\rangle$, but if it acts on $|\beta\rangle$, we get *something proportional to* $|2\beta\rangle$, since we can have multiple bosons in the same state. The key to figuring out this proportionality factor is to demand that $a_\beta{}^\dagger a_\beta$ acts as the **number operator** for the state β. So

$$a_\beta{}^\dagger a_\beta |n_1, n_2, \ldots, n_\beta, \ldots\rangle = n_\beta |n_1, n_2, \ldots, n_\beta, \ldots\rangle .$$

From the relation

$$\langle n_1, n_2, \ldots, n_\beta, \ldots| a_\beta{}^\dagger a_\beta |n_1, n_2, \ldots, n_\beta, \ldots\rangle = n_\alpha,$$

we find

$$a_\beta |n_1, n_2, \ldots, n_\beta, \ldots\rangle = \sqrt{n_\beta} |n_1, n_2, \ldots, n_\beta - 1, \ldots\rangle .$$

Now, we can determine the proportionality factor for $a_\beta{}^\dagger$. Taking

$$a_\beta a_\beta{}^\dagger |n_1, n_2, \ldots, n_\beta, \ldots\rangle = \sqrt{n_\beta + 1} c |n_1, n_2, \ldots, n_\beta, \ldots\rangle ,$$

where c is the proportionality factor to be determined. Hit this again with $a_\beta{}^\dagger$ to find

$$a_\beta{}^\dagger a_\beta a_\beta{}^\dagger |n_1, n_2, \ldots, n_\beta, \ldots\rangle = \sqrt{n_\beta + 1} c^2 |n_1, n_2, \ldots, n_\beta + 1, \ldots\rangle .$$

We can rewrite the LHS of this equation by noting that the first two operators are the number operator, so that

$$a_\beta{}^\dagger a_\beta a_\beta{}^\dagger |n_1, n_2, \ldots, n_\beta, \ldots\rangle = (n_\beta + 1) c |n_1, n_2, \ldots, n_\beta + 1, \ldots\rangle .$$

Thus,

$$c = \sqrt{n_\beta + 1}.$$

This immediately gives the desired relation, since it is that satisfied by the harmonic oscillator algebra. Whew!

The last identity follows since interchange of the two bosonic states which we create will not force a sign change, so the commutator will vanish.

Exercise 2. Helium.

a) Consider a singly-ionized helium ion. How much more energy does it take to ionize its bound electron compared to hydrogen?
Hint: Use dimensional analysis and the fact that the ground state energy for hydrogen is

$$E_0 = -13.6 \text{ eV} \sim -\alpha^k mc^2,$$

where $\alpha \sim 1/137$ is the fine structure constant (a dimensionless constant formed from $e,\ \hbar$, and c) and k is an integer that you should determine.

b) Still with He^+. What is the wavelength of the emitted photon during the electron transition from $n = 2 \to 1$?
Hint: $hc = 1240$ eV $\cdot$ nm. This formula is so useful that you should memorize it!!!

c) Now, consider the usual helium-4. Which ground state has higher energy, parahelium (spin singlet) or orthohelium (spin triplet)? Why? ***Griffiths 5.14.*** How would this change if the two electrons are identical bosons?

a) The only difference between singly-ionized helium and hydrogen is the number of protons. The energy levels for hydrogen-like atoms come from a potential in the Schrödinger equation which is proportional to Ze^2, where Z is the number of protons. We need to figure out how the energy scales as a function of Z. Dimensional analysis gives that

$$\alpha = \frac{e^2}{\hbar c},$$

while

$$13.6 \text{ eV} \sim \alpha^2 mc^2,$$

since $mc^2 = 511$ keV. Since E scales as e^4, it also scales as Z^2. Thus, helium-3 is 4 times harder to ionize than hydrogen.

b) Recall that $E = pc$ for light; hence,

$$E = \frac{hc}{\lambda}.$$

Since $E_2 - E_1 = 3E_0$, we have

$$\lambda = \frac{hc}{3E_0}.$$

Since $hc = 1240$ eV $\cdot$ nm, we have

$$\lambda \approx 30 \text{ nm}.$$

c) Since the triplet is symmetric, the spatial wavefunction which is associated to orthohelium must be antisymmetric. Since the lowest energy state is symmetric (both electrons in $n = 1$), it follows that the triplet has higher energy than the singlet. If they were both bosons, then we'd have the opposite.