Week 3 Worksheet
Identical Particles Continued (and Helium)

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Exercise 1. Symmetries of Many-Particle States.

a) Consider a system of two identical particles. Define the operator P_{12} via

$$P_{12} |a\rangle |b\rangle = |b\rangle |a\rangle .$$

Show that ${P_{12}}^2 = \mathbb{1}$, the identity operator, and that the eigenvalues of P_{12} are ± 1. Thus, show that its eigenvectors are either totally symmetric or antisymmetric. We call P_{12} a **permutation operator**. In this case, there are only two such operators: P_{12} and ${P_{12}}^2 = \mathbb{1}$.

b) Generalize part (a) to systems of three identical particles. You should find that you have *six* permutation operators (note that the identity is a permutation operator). Assuming the hamiltonian is invariant under each of these operators, is there a complete set of common eigenvectors?

c) ***Griffiths 5.8.*** In the situation of (b), suppose that the particles have access to three distinct one-particle states, $|a\rangle$, $|b\rangle$, and $|c\rangle$. For example, $|abc\rangle$ is an allowed state, as is $|aaa\rangle$. How many states can be constructed if they are (i) bosons or (ii) fermions?

d) Suppose we have a single-particle fermion state $|\alpha\rangle$ and a single-particle bosonic state $|\beta\rangle$. Just like for the harmonic oscillator, we can define **creation operators** ${C_\alpha}^\dagger$ and ${a_\beta}^\dagger$, such that given any state $|\psi\rangle$,

$$\begin{aligned} {C_\alpha}^\dagger |\psi\rangle &= |\alpha\psi\rangle \\ {a_\beta}^\dagger |\psi\rangle &= |\beta\psi\rangle . \end{aligned}$$

The operators $C_\alpha{}^\dagger$ and $a_\beta{}^\dagger$ have the following properties. You don't need to prove them.

$$\begin{aligned} C_\alpha \left|\alpha\psi\right\rangle &= \left|\psi\right\rangle \\ a_\beta \left|\beta\psi\right\rangle &= \left|\psi\right\rangle \\ C_\alpha \left|0\right\rangle = a_\beta \left|0\right\rangle &= 0 \\ C_\alpha{}^\dagger C_\alpha{}^\dagger &= 0 \\ \{C_\alpha, C_{\alpha'}{}^\dagger\} \equiv C_\alpha C_{\alpha'}{}^\dagger + C_{\alpha'}{}^\dagger C_\alpha &= \delta_{\alpha\alpha'} \mathbb{1} \\ \{C_\alpha{}^\dagger, C_{\alpha'}{}^\dagger\} &= 0 \\ [a_\beta, a_{\beta'}{}^\dagger] &= \delta_{\beta\beta'} \mathbb{1} \\ [a_\beta{}^\dagger, a_{\beta'}{}^\dagger] &= 0, \end{aligned}$$

where $\left|0\right\rangle$ denotes a state with no particles at all. To what extent is a bound pair of fermions equivalent to a boson?

Hints: Use the symmetries of many-particle states and the (anti-)commutation relations of the creation/annihilation operators constructed in parts (a)-(d). What algebra must the creation/annihilation operators for the bound pair satisfy? In particular, you should show that

$$[D_{12}, D_{12}{}^\dagger] = \mathbb{1} - C_1 C_1{}^\dagger - C_2 C_2{}^\dagger,$$

where $D_{12}{}^\dagger$ is the creation operator for a bound pair of fermions in states 1 and 2, respectively.

e) **Challenge.** Prove the properties given in (d).
Hints: It may be useful to use the notation $\sim \alpha$ for the α "orbital" being *unoccupied*. To show the first relation for C_α, try to first show that $C_\alpha \left|\alpha\right\rangle = \left|0\right\rangle$. For the anti-commutator relations, consider separately the cases $\alpha \neq \alpha'$ and whether the α or α' orbitals are occupied.

Exercise 2. Helium.

a) Consider a singly-ionized helium ion. How much more energy does it take to ionize its bound electron compared to hydrogen?
Hint: Use dimensional analysis and the fact that the ground state energy for hydrogen is

$$E_0 = -13.6 \text{ eV} \sim -\alpha^k mc^2,$$

where $\alpha \sim 1/137$ is the fine structure constant (a dimensionless constant formed from e, $\hbar$, and c) and k is an integer that you should determine.

b) Still with He^+. What is the wavelength of the emitted photon during the electron transition from $n = 2 \to 1$?
Hint: $hc = 1240$ eV $\cdot$ nm. This formula is so useful that you should memorize it!!!

c) Now, consider the usual helium-4. Which ground state has higher energy, parahelium (spin singlet) or orthohelium (spin triplet)? Why? ***Griffiths 5.14.*** How would this change if the two electrons are identical bosons?