Review Session Problems 2

Jacob Erlikhman

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Exercise 1. When we solve the hydrogen atom, we assume that the nucleus is a point charge. In this problem, we will compute the approximate change to the energy levels due to the finite size of the nucleus. This is called the **volume effect**. Model the nucleus as a uniform sphere of radius $r_0 A^{1/3}$, where $A^{1/3}$ is the number of nucleons (so this works for e.g. deuterium) and $r_0 = 1.3 \cdot 10^{-13}$ cm.

a) What is the potential $V(r)$?
Hint: Outside the nucleus, $V(r)$ is just the Coulomb potential. Inside the nucleus, use Gauss' law to determine $V(r)$.

b) What is H', where H^0 is the hydrogen atom hamiltonian?

c) Argue that the $\ell > 0$ states are only slightly affected by this perturbation.
Hint: Think about the small r behavior of the wavefunctions for s-states vs. $\ell > 0$ states.

d) Calculate the correction to the energy levels for all states with $\ell = 0$. Note that

$$R_{n0}(0) = \frac{2}{(na_0)^{3/2}},$$

where $a_0 = \hbar^2/me^2$.

e) For hydrogen, calculate the correction to the $n = 1$ and $n = 2$ states in eV.

f) Fine structure is of order $\alpha^4 mc^2$. Compare the magnitude of the volume effect to that of fine structure.

Exercise 2. Explain the physical origins of

a) fine structure

b) Lamb shift

c) hyperfine structure.

Exercise 3. *Griffiths 8.19* Find the lowest bound on the ground state of hydrogen using the variational principle and an exponential trial wavefunction,

$$\psi(\mathbf{r}) = Ae^{-br^2},$$

where A is determined by normalization and b is a variational parameter. Express your answer in eV.

Exercise 4. *Griffiths 9.18* When we turn on an external electric field, it should be possible to ionize the electron in an atom. A crude model for this is to suppose that a particle is in a very deep, one-dimensional finite square well.

a) What is the energy of the ground state, measured up from the bottom of the well? Assume that $V_0 \gg \hbar^2/ma^2$.

b) Introduce the perturbation $H' = -\alpha x$, where $\alpha \equiv eE_{\text{ext}}$. Assume that $\alpha a \ll \hbar^2/ma^2$, and sketch the total potential, noting that the electron can tunnel out in the direction of positive x.

c) Calculate

$$\gamma = \frac{1}{\hbar} \int |p(x)| \, \mathrm{d}x,$$

and estimate the time it would take for the particle to escape,

$$\tau = \frac{2x_1}{v} e^{2\gamma},$$

where x_1 is the distance the electron must travel to reach the tipping point of the potential and v is the speed of the electron.

d) Plug in some numbers, e.g. $V_0 = 20$ eV, $E_{\text{ext}} = 7 \cdot 10^6$ V/m, $a = 10^{-10}$ m. Calculate τ, and compare it to the age of the universe.