Week 2 Worksheet
137A Review; QM in 3D (and some spin)

Jacob Erlikhman

1/16/24

Exercise 0. Warm up.

a) What is the momentum operator in the position representation in three dimensions?
Hint: What is the canonical commutation relation?

b) From (a), what is $p^2 = \mathbf{p} \cdot \mathbf{p}$ in three dimensions in the position representation?

c) Starting from the time-dependent Schrödinger equation in three dimensions for a potential $V(\mathbf{x})$, derive the time-independent version.
Hint: Use separation of variables.

Exercise 1. Rigged Hilbert Space. A Hilbert space is mathematically defined as a *complete* vector space with an inner product. A vector space with an inner product is **complete** if it includes not only all finite sums of vectors in a basis, but also all limits of convergent sequences, i.e. given a sequence (v_n) of vectors in the Hilbert space, v is the **limit** of the sequence if $\lim_{n\to\infty} \|v_n - v\| = 0$, where $\|v\| = \sqrt{v \cdot v}$.

a) Consider a Hilbert space $\mathcal{H}$ that consists of all functions $\psi(x)$ such that

$$\int_{-\infty}^{\infty} |\psi(x)|^2 \,\mathrm{d}x < \infty.$$

Show that there are functions in $\mathcal{H}$ for which $\hat{x}\psi(x) = x\psi(x)$ is not in $\mathcal{H}$.

b) Consider the function space Ω in $\mathcal{H}$ which consists of all $\varphi(x)$ that satisfy the set of conditions

$$\int_{\infty}^{\infty} |\varphi(x)|^2 (1 + |x|)^n \,\mathrm{d}x < \infty,$$

for any $n \in \{0, 1, 2, \ldots\}$. Show that for any $\varphi(x)$ in Ω, $\hat{x}\varphi(x)$ is also in Ω. Ω is called the **nuclear** space.
Hint: Binomial theorem.

c) The **extended** space $\Omega^\times$ consists of those functions $\chi(x)$ which satisfy

$$(\chi, \varphi) = \int_{-\infty}^{\infty} \chi^*(x)\varphi(x) \,\mathrm{d}x < \infty,$$

for any φ in Ω, where (,) is the inner product on $\mathcal{H}$. Which of the following functions belong to Ω, to $\mathcal{H}$, and/or to $\Omega^\times$?
Hints: Note that in order to sit in Ω, functions must vanish faster than any power of x as $|x| \to \infty$. Thus, as long as functions don't diverge at ∞ more strongly than any power of $|x|$, they are in $\Omega^\times$.

Remark. The collection $(\Omega, \mathcal{H}, \Omega^{\times})$ is called "rigged Hilbert space," and this is a rigorous way to include all the formalism (e.g. eigenvectors of position are delta functions, and hence can't belong to an L^2 space) into the Hilbert space formulation of quantum mechanics. Note that $\Omega \subset \mathcal{H} \subset \Omega^{\times}$ (it's easy to see this once you realize $\mathcal{H} = \mathcal{H}^{\times}$). For more details, see Ballentine *Quantum Mechanics*, Chapter 1.

i) $\sin(x)$

ii) $\sin(x)/x$

iii) $x^2 \cos(x)$

iv) e^{-ax}, $a > 0$.

v) $\dfrac{\ln(1 + |x|)}{1 + |x|}$

vi) e^{-x^2}

vii) $x^4 e^{-|x|}$

viii) $\delta(x - a)$ for a real.

Exercise 2. Harmonic Oscillator. Solve the eigenvalue problem for the 3-D isotropic, harmonic oscillator, whose hamiltonian is $H = p^2/2m + m\omega^2 x^2/2$, where $p^2 = \mathbf{p} \cdot \mathbf{p}$, $x^2 = \mathbf{x} \cdot \mathbf{x}$ is the 3-D dot product.
Hint: There's a way to do this without any calculations (if you remember the 1-D oscillator)!

Exercise 3. A particle of mass m is placed in a finite spherical well

$$V(r) = \begin{cases} -V_0, & r \leq a \\ 0, & r \geq a \end{cases}.$$

Find the equation that quantizes the energy (you don't need to solve it), by solving the radial Schrödinger equation with $\ell = 0$. Explain how you could solve this equation and obtain the energies. Show that there is no bound state if $V_0 a^2 < \pi^2 \hbar^2 / 8m$.
Hint: Recall that the radial Schrödinger equation is identical to the time-independent, 1-dimensional Schrödinger equation with the wavefunction replaced by $u(r) = rR(r)$ (where $\psi(r, \theta, \varphi) = R(r)\Theta(\theta)\Phi(\varphi)$) and potential $V_{\text{eff}}(r) = V(r) + \frac{\hbar^2 \ell(\ell+1)}{2mr^2}$.

Exercise 4. Spin Representations. a) Find the eigenvalues and eigenvectors of S_z.

b) Do the same for S_y, and write them in terms of $|\uparrow\rangle$ and $|\downarrow\rangle$, the eigenvectors of S_z.

c) For a system of two spin 1/2 particles, starting with the "highest weight" state $|\uparrow\uparrow\rangle$, find all the states in the triplet.
Hint: Apply the lowering operator.

d) For a system of two spin 1/2 particles, are there any other states than the ones you found in (c)? If so, what are they? What is the action of S_-, S_+ on them?

e) Describe how you would approach finding the Clebsch-Gordan coefficients for arbitrary spin systems.