Week 8 Worksheet
More Perturbation Theory

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Exercise 1. ***Griffiths 7.54.*** Last week, you derived the first order correction to the expectation value of an observable A in the n^{th} energy eigenstate of a system perturbed by H'. You found

$$\langle A\rangle^1 = 2\sum_{m\neq n}\frac{\left\langle\psi_n^0\middle|A\middle|\psi_m^0\right\rangle\left\langle\psi_m^0\middle|H'\middle|\psi_n^0\right\rangle}{E_n^0-E_m^0}.$$

Suppose we have a particle of charge q in a weak electric field $\mathbf{E} = E_{\text{ext}}\hat{x}$, so that $H' = -qE_{\text{ext}}x$. This induces a dipole moment $p_e = qx$ in the "atom." The expectation value of p_e is proportional to the applied field, and the proportionality factor is called the **polarizability**, α. Show that

$$\alpha = -2q^2\sum_{m\neq n}\frac{\left|\langle\psi_n^0|x|\psi_m^0\rangle\right|^2}{E_n^0-E_m^0}.$$

Find α for the ground state of a 1-D harmonic oscillator, and compare the classical answer.

Hint: Recall that x can be written in terms of creation and annihilation operators. Given

$$H^0 = \frac{1}{2m}\left[p^2+(m\omega x)^2\right],$$

you can derive what a and $a^\dagger$ should be in terms of x and p by using the sum of squares formula. To get the "usual" form, rescale each of them by $a \to \frac{1}{\sqrt{\hbar\omega}}a$ (so that the hamiltonian can be written $\frac{H^0}{\hbar\omega} = a^\dagger a + 1/2$).

Exercise 2. ***Griffiths 7.45.*** **Stark Effect in Hydrogen.** When an atom is placed in a uniform electric field $\mathbf{E}_{\text{ext}}$, the energy levels are shifted. This is known as the **Stark effect**. You'll analyze the Stark effect for the $n = 1$ and $n = 2$ states of hydrogen. Suppose $\mathbf{E}_{\text{ext}} = E_{\text{ext}}\hat{z}$, so that

$$H' = eE_{\text{ext}}r\cos\theta$$

is the perturbation of the hamiltonian for the electron, where $H^0 = \frac{p^2}{2m} - \frac{e^2}{4\pi\varepsilon_0}\frac{1}{r}$.

a) Show that the ground state energy is unchanged at first order.

b) How much degeneracy does the first excited state have? List the degenerate states.

c) Determine the first-order corrections to the energy. Into how many levels does E_2 split?

Hint: All W_{ij} are 0 except for two, and you can avoid doing all of the zero integrals in this problem by using symmetry and selection rules. You'll need the following

$$\psi_{210} = \frac{1}{2\sqrt{6}} a^{-3/2} \frac{r}{a} e^{-r/2a} \sqrt{\frac{3}{4\pi}} \cos\theta$$
$$\psi_{200} = \frac{1}{\sqrt{2}} a^{-3/2} \left(1 - \frac{r}{2a}\right) e^{-r/2a} \frac{1}{2\sqrt{\pi}}.$$

d) What are the "good" wavefunctions for (b)? Find the expectation value of the electric dipole moment in each of these states.